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A MOTION SYNTHESIS APPROACH TO SOLVING ALT-BURMESTER PROBLEMS BY EXPLOITING FOURIER DESCRIPTOR RELATIONSHIP BETWEEN PATH AND ORIENTATION DATA

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ABSTRACT

This paper presents a generalized framework to solve m-pose, n-path points mixed synthesis problems, known as Alt-Burmester problems, using a task-driven motion synthesis approach. We aim to unify the path and motion synthesis problems into an approximate mixed synthesis framework. Fourier descriptors are used to establish a closed-form relationship between the path and orientation data. This relationship is then exploited to formulate mixed synthesis problems into pure motion synthesis ones. We use an efficient algebraic fitting based motion synthesis algorithm to enable simultaneous type and dimensional synthesis of planar four-bar linkages.

keywords: *Alt-Burmester, Mixed Synthesis, Path Synthesis, Motion Synthesis, Fourier Descriptors, Planar Four-bar linkage, Type and Dimensional Synthesis*

1 Introduction

Conventionally, mechanism synthesis problems have been categorized and studied independently as path, motion, and function synthesis problems [1]. Path synthesis problems specify only path-point coordinates (x_i, y_i), while motion synthesis specify only pose constraints (x_j, y_j, ζ_j). In function synthesis, only input-output angle sets (θ_k, ψ_k) are specified. Unfortunately, most of the real world problems do not conform to such a rigid categorization – many practical problems provide a mixture of path, motion and function synthesis requirements. However, a synthesis approach which fluidly incorporates all the three conventional synthesis problems has been elusive. As a result, designers have to often compromise on design specifications. In this paper, the focus is on synthesis of planar four-bar mechanisms for a hybrid of path and motion synthesis problems. This problem formulation, which consolidates both path and orientation data, has been termed as mixed synthesis in this paper.

Prof. Murray's group termed the combined path and motion problems as the Alt-Burmester problems [2] named after Alt's [3] and Burmester's [4] work on path and motion generation, respectively. Brake et al. [5] discuss the dimensionality of solution sets for a variety of path-point and pose combinations. However, a finite number of solutions exist only for a subset of possible *m*-pose, *n*-path point synthesis problem. For example, there exist finite solutions for nine path-points and for five poses independently. We define such problems to be fully constrained problems. For lesser number of path-points or poses, usually an infinite number of solutions are obtained. Subsequently, the authors explore only fully-constrained or under-constrained problem sets where up to nine constraints can be used to find four bar mechanism parameters. This ignores the vast majority of over-constrained problems in *m*-pose, *n*-path point mixed synthesis family of problems, where exact solutions are not possible and only approximate, albeit useful solutions, can still be obtained. This is reflective of real-world design problems, which usually impose a large number of often challenging constraints.

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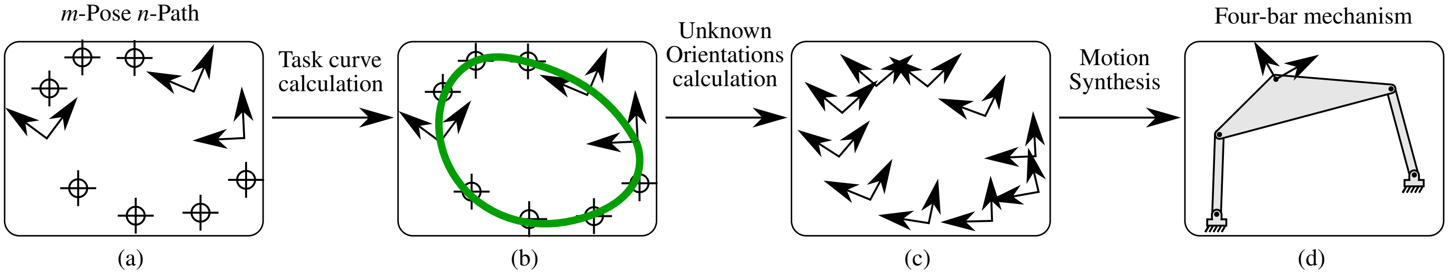

FIGURE 1: An Overview of our Approach to the Alt-Burmester Problems: (a) specify m-pose, n-path points; (b) a task curve is fit through the $m+n$ path points using Fourier series; (c) use the harmonic content of the path data to find the *missing* orientations at the n-path points; and (d) finally, compute both type and dimensions of planar four-bar linkages.

A graphical approach has been presented by Zimmerman [6] to solve the mixed path, motion and function problem using sketcher tool built in modern CAD softwares. The proposed methodology can conveniently solve under-constrained and fully-constrained mixed synthesis problems and generate four-bar mechanisms. The advantage of this approach is that it comes close to unifying path, motion and function synthesis. However, this methodology is unable to solve generalized m pose, n path-point synthesis problems, which may be over-constrained.

Motion synthesis turns out to be a mathematically less complex problem than path synthesis as each dyad can be calculated independently effectively halving the number of unknowns in the equations. Typically, path synthesis problems involve solving a nonlinear system of equations. We have recently presented a generalized framework for solving motion synthesis problems with a fast algorithm that solves a linear system of equations using singular value decomposition [7, 8, 9, 10, 11]. The algorithm produces multiple solutions and can compute both the type and dimensions of the four-bar mechanisms. In this paper, we are presenting an approach to solve Alt-Burmester problems by reducing it to pure motion synthesis problems so that the aforementioned algorithm can be leveraged. In a planar four-bar linkage, path of a coupler point is inextricably tied to the orientation of the coupler. This coupling can be revealed by analyzing the harmonic content of the path and orientation data. First, an analytical relation between the orientation data and path-point data using the harmonic breakdown of the loop closure equation is calculated. Then, this relation is used to reformulate the mixed synthesis problem into a motion synthesis problem by attaching adequate orientations to each input path point and consequently turning them into poses. The Fourier approximation based analytical approach proposed in this paper can handle almost all possible variations of path points or poses. Once, the problem has been converted into a pure motion synthesis problem, we re-purpose our algebraic fitting approach to solve for four-bar linkages. Fig. 1 provides an overview of this approach.

We note that here mixed synthesis does *not* refer to the mixed exact-approximate path or motion synthesis, where we have a mix of exact and approximate constraints. Our definition of *mixed* refers to a mixture of path-point and pose constraints.

This paper makes original contribution on three fronts: 1) the formulation of a Fourier descriptor based closed form relationship between coupler orientations and path, 2) the novel use of this relationship to solve the generalized m-pose, n-path point mixed synthesis problem, and 3) the incorporation of task-driven algebraic fitting based motion synthesis within the mixed synthesis algorithm for synthesis of planar dyads. We note that the simultaneous type and dimensional synthesis of planar mechanisms is enabled by the algorithm presented in [11] and this paper retains that aspect of synthesis.

Rest of the paper is organized as follows. Section 2 calculates a new path-orientation formulation from existing four-bar loop closure Fourier decomposition. Section 3 discusses the use of path-orientation relationship to reformulate mixed synthesis into motion synthesis problems. Section 4 reviews algebraic fitting based motion synthesis algorithm. Section 5 proposes the new algorithm to solve mixed synthesis problem. In the end, in section 6, we present a few examples of mixed synthesis problems.

2 Fourier Descriptors based relations

Use of Fourier descriptors is abundant in the domain of mechanism synthesis. It has been used for planar four bar mechanism synthesis using optimization routines [12, 13, 14, 15, 16], atlas-based search algorithms [17, 18] and machine learning approach [19]. Fourier descriptors have also been used to synthesize spherical [20] and spatial mechanism [21]. A class of single degree of freedom open-loop mechanisms termed as planar coupled serial chain mechanisms [22, 23] can be generated with the help of Fourier descriptors.

In this section, we are interested in exploring the relationship between the coupler path and coupler orientation to establish a closed form relationship between them. This would give us a framework for dealing with both pose and path constraints simultaneously. Path and motion synthesis formulations which use

Fourier decomposition of four-bar closure equation [14, 15, 16] are used as a starting point here. In [16], Li et al. presented a decomposition of the design space of four-bar mechanisms by using Fourier descriptors in the context of planar motion approximation.

A four-bar mechanism is represented by its design parameters x_0, y_0, l_1, l_2, l_3, l_4, r, θ_1, and α as displayed in Fig. 2. Harmonic decomposition of four-bar loop closure equation has been analyzed to independently fit rotational and translational Fourier descriptors. Coupler angle λ represents the orientation of coupler link with respect to fixed link at any given instant. Point P is the location of the coupler point in the global frame. Coupler orientation ζ refers to the orientation of end effector attached at the coupler point at any time. δ and λ are two additional angles defined with reference to θ_1 as shown in figure. Our goal is to find an explicit closed form relationship between coupler path and orientation which forms the heart of our mixed synthesis algorithm.

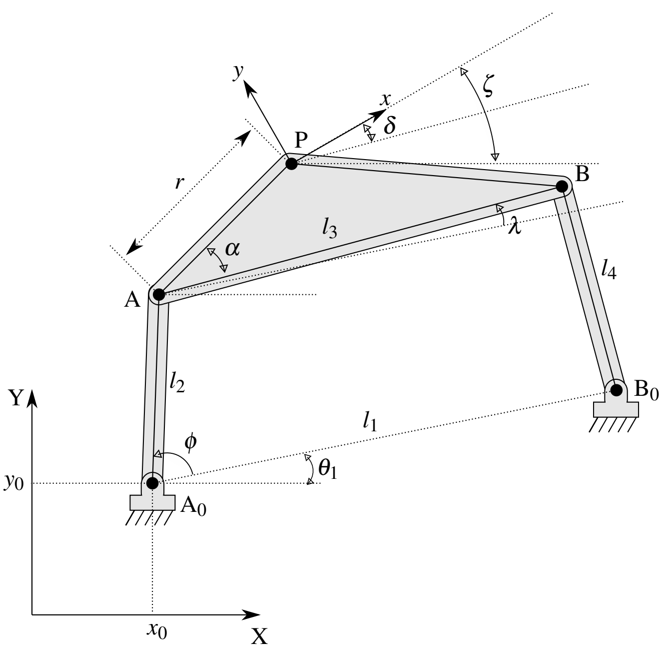

FIGURE 2: Visualization of parameters describing a four-bar mechanism

2.1 Coupler angle

The Fourier series representation of the coupler angle (λ) for a four bar mechanism is given as

$$e^{j\lambda} = \sum_{k=-\infty}^{\infty} C_k e^{jk\phi} = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega t} e^{jk\phi_0}, \tag{1}$$

where C_k are the harmonic descriptors of coupler angle. Here, ϕ is the crank angle, ϕ_0 the initial crank angle, and ω is the constant angular speed of the input link.

2.2 Coupler path

The analytic equation which defines the path (P) of coupler point for a four bar mechanism is

$$P = A_0 + l_2 e^{j\theta_1} e^{j\phi} + r e^{j\alpha} e^{j\theta_1} e^{j\lambda}, \tag{2}$$

where A_0 is the position of actuator fixed pivot, l_2 is the length of actuator link, θ_1 is the angle of fixed link, and r and α are the coupler parameters. Being a periodic function, it can also be represented as a Fourier series:

$$\mathbf{P} = \sum_{k=-\infty}^{\infty} P_k e^{jk\omega t}. \tag{3}$$

Substituting (1) into (2) and then equating resulting (2) and (3), we get harmonic descriptors P_k for path as following:

$$P_0 = C_0 r e^{j(\alpha+\theta_1)} + (jy_0 + x_0); \qquad k = 0, \tag{4}$$

$$P_1 = C_1 e^{j\phi_0} r e^{j(\alpha+\theta_1)} + l_2 e^{j\theta_1} e^{j\phi_0}; \qquad k = 1, \tag{5}$$

$$P_k = C_k e^{jk\phi_0} r e^{j(\alpha+\theta_1)}; \qquad k \neq 0, 1. \tag{6}$$

2.3 Coupler orientation

The orientation (ζ) at the coupler point for a four bar mechanism can be defined as

$$\zeta = \delta + \lambda + \theta_1 = \arg(e^{j(\delta+\lambda+\theta_1)}), \tag{7}$$

where δ is the fixed angle at which moving frame is attached to coupler with respect to $\theta_1 + \lambda$. As λ varies periodically while δ and θ_1 remain constant, the orientation can be decomposed harmonically as

$$e^{j(\delta+\lambda+\theta_1)} = \sum_{k=-\infty}^{\infty} C_k^* e^{jk\omega t} \tag{8}$$

where C_k^* are the harmonic descriptors for orientation and defined as

$$C_k^* = C_k e^{j(\delta+\theta_1)} e^{jk\phi_0}. \tag{9}$$

2.4 Path-orientation relation

With the above relations, it is now possible to find explicit closed form relations between the fourier descriptors of coupler path and coupler orientation data. Using Eqns. (4), (5), (6), and (9), relationship between the harmonic descriptors of path (P_k) and orientation (C_k^*) is found to be

$$C_0^* = (P_0 + z_2)z_1, \tag{10}$$
$$C_1^* = (P_1 + z_3)z_1, \tag{11}$$
$$C_k^* = P_k z_1, \tag{12}$$

where

$$z_1 = \frac{e^{j(\delta-\alpha)}}{r}, \tag{13}$$
$$z_2 = -(x_0 + jy_0), \tag{14}$$
$$z_3 = -(l_2 e^{j\theta_1} e^{j\phi_0}). \tag{15}$$

Using the above relationship, the orientation at coupler point can be defined exclusively using path harmonic descriptors as follows

$$e^{j\zeta(t)} = z_1 \left(z_2 + z_3 e^{j\omega t} + \sum_{k=-\infty}^{\infty} P_k e^{jk\omega t} \right). \tag{16}$$

Subsequently, using Eq. (16) for n path points, the system of equation describing orientation at each path point turns out to be

$$\begin{bmatrix} e^{j\zeta_1} \\ e^{j\zeta_2} \\ \vdots \\ e^{j\zeta_n} \end{bmatrix} = \begin{bmatrix} 1 & e^{j\omega t_1} & \sum_{k=p}^{-p} P_k e^{jk\omega t_1} \\ 1 & e^{j\omega t_2} & \sum_{k=p}^{-p} P_k e^{jk\omega t_2} \\ \vdots & \vdots & \vdots \\ 1 & e^{j\omega t_n} & \sum_{k=p}^{-p} P_k e^{jk\omega t_n} \end{bmatrix} \begin{bmatrix} z_1 z_2 \\ z_1 z_3 \\ z_1 \end{bmatrix}. \tag{17}$$

Thus, the orientations at different points of a four bar coupler path are dependent on path descriptors and three complex variables z_1, z_2, and z_3, which are termed as Mixed Synthesis Parameters (MSP). The MSP are dependent on four-bar mechanism design parameters according to Eqns. (13), (14), and (15). Eq. (17) is key to the mixed synthesis formulation. It will help us find orientation information for path points as discussed in the next section.

3 Calculating unknown orientations

The aim of this section is to reformulate m-pose, n-path mixed synthesis problems into a $m+n$-pose motion synthesis problems. To enable that, generation of orientation data for n path points and converting them to n poses is required. Eq. (17) will be used to accomplish this objective.

First, for a general m-pose, n-path mixed synthesis problem, a smooth task path with low harmonic Fourier descriptors (T_k) passing through $m+n$ points can be written as

$$T \approx \sum_{k=-p}^{p} T_k e^{ik\Phi}. \tag{18}$$

To calculate descriptors for uniform phase parametrization, inverse discrete Fourier transform can be calculated. Even open path curves can be incorporated using a least square fitting approach which can be formulated as minimization of

$$\Delta = \sum_{i=1}^{n} \left\| z(t_i) - \sum_{k=-p}^{p} \alpha_k e^{ik\omega_o t_i} \right\|^2, \tag{19}$$

where Δ is the error measure and $z(t_i)$ are the complex-valued point data at time t_i.. Analytically solving the least square problem gives a linear system of equation as follows

$$\Omega \mathbb{X} = \mathbb{Y}, \tag{20}$$

where

$$\mathbb{X} = [\ldots, \underset{m\rightarrow}{\alpha_m}, \ldots]^T, \tag{21}$$

$$\Omega = \begin{bmatrix} & \cdots & \\ \vdots & \sum_{i=0}^{n} e^{i(k-m)\theta_i} & \vdots \\ & \cdots & \end{bmatrix}_{k\rightarrow} \downarrow m, \tag{22}$$

$$\mathbb{Y} = [\ldots, \underset{m\rightarrow}{\sum_{i=0}^{n} z(t_i) e^{-im\theta_i}}, \ldots]^T. \tag{23}$$

Here, k and m vary from $-p$ to p which denote the column and row index of an element in the matrix. LU decomposition can be used to solve the above system. More details can be found in the work done by Wu et al. [14]. In a companion paper [24], we have proposed a method to calculate optimal parametrization for task curve for Fourier descriptor fitting of the path data. In our implementation, task curves are represented using five descriptors i.e. $p \in [-2,2]$.

The reasoning behind using a low harmonic task curve is supported in literature [16,25], which says that the magnitude of high harmonics for coupler path of a four-bar mechanism has an insignificant impact. Thus, the fitted task path is a good prospective four bar coupler curve.

The intention now is to find the values of MSP i.e. $\{z_1, z_2, z_3\}$ using available orientation data and subsequently generate unknown orientations. There are multiple ways the relation given in Eq. (17) can be used to convert mixed synthesis into motion synthesis problem depending on the MSP fitting problem being under-constrained, fully-constrained or over-constrained.

For a fully-constrained MSP fitting problem, three poses are required to calculate the MSP directly from Eq. (17). Physically, this condition makes perfect sense as the user might know orientations at the initial position, final position and an additional intermediate location while a sequence of path points might be given in addition.

For under-constrained MSP fitting problem, there are only two or one poses. As a result additional constraints are required to uniquely calculate the MSP. The MSP are dependent on four-bar mechanism parameters according to Eqns. (13), (14), and (15). These equations can be used to generate additional constraints which are called Mixed Constraints (MIC). The three possible MIC are

1. Specify actuating fixed pivot i.e. $\{x_0, y_0\}$
2. Specify coupler parameters i.e. $\{r, \alpha, \delta\}$
3. Specify scale of input link, orientation of fixed pivot line, and initial angle i.e. $\{l_2, \theta_1, \phi_0\}$

Thus, if two poses are input by the user, one MIC is required to fully define the system of equations in Eq. (17). If only one pose is specified by user, two MIC are required to solve the problem. Two pose problem is fairly common when only the first and last orientations are important, such as in pick-and-place operations. The MIC also mirror practical user-specified constraints, such as selection of the location of the fixed pivot where an actuator might be situated. In another case, there might be a restriction on coupler link dimensions. Thus, the MIC represent a set of practical design constraints, which provide designers useful tools.

It is important to note that a pure path synthesis problem cannot be restructured into a motion synthesis problem without fully defining all three MSP. However, constraining all MSP simultaneously makes the synthesis trivial as by fixing all MIC, the four-bar linkage is already fixed.

For over-constrained MSP fitting problems, the number of poses specified is more than three. In this case, a least square solution to Eq. (17) can be calculated using complex Singular Value Decomposition (SVD). Real SVD solvers, which are more easily available, can also be used by reducing the complex system of equation in Eq. (17) into an equivalent real system of equation in accordance with [26]. The K_1 formulation presented in [26] has been used in our implementation. According to the formulation, a complex system of equation

$$(A+iB)(x+iy) = b+ic \tag{24}$$

can be written as a real system of equation

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix} \tag{25}$$

Finding least square solution to this equivalent real system of equations gives the solution to original complex problem and values of MSP can easily be calculated in over-constrained cases.

Once the values of MSP z_1, z_2, z_3 are calculated using m poses, orientations at n path points can be found out by simple matrix multiplication using the system of equation in Eq. (17). As a result, n path points and m poses are converted to $m+n$ poses. The motion synthesis algorithm can now be used to calculate dyads. A review of algebraic fitting based motion synthesis algorithm is discussed in next section.

4 Motion synthesis algorithm

Now that the mixed synthesis problem has been reformulated as motion synthesis problem, solution mechanisms can be achieved by calculating the dyads. Algebraic fitting based motion synthesis algorithm [7,8,9,10] has been used in our implementation. The goal of this approach is to map the poses into image space and fit geometric constraint manifolds described using image space coordinates to calculate the least square solution dyads.

This approach enables us to simultaneous carry-out type and dimensional synthesis of four-bar linkages i.e. it takes into consideration the possibility of both revolute and prismatic joints. Another benefit of the approach is its fast and efficient computation.

First, using kinematic mapping, each of the user-defined pose $\{d_1, d_2, \phi\}$ is mapped to quaternion space defined by a four-dimensional vector $\mathbf{Z} = \{Z_1, Z_2, Z_3, Z_4\}$. This quaternion space is also termed as the Image Space. The relations mapping real space to image space are

$$Z_1 = \frac{1}{2}\left(d_1 \cos\frac{\phi}{2} + d_2 \sin\frac{\phi}{2}\right), \tag{26}$$

$$Z_2 = \frac{1}{2}\left(-d_1 \sin\frac{\phi}{2} + d_2 \cos\frac{\phi}{2}\right), \tag{27}$$

$$Z_3 = \sin\frac{\phi}{2}, \tag{28}$$

$$Z_4 = \cos\frac{\phi}{2}. \tag{29}$$

The geometric motion constraint which every dyad needs to satisfy is given by the G-constraint manifold described as

$$\begin{aligned} q_1(Z_1^2+Z_2^2)+q_2(Z_1Z_3-Z_2Z_4)+q_3(Z_2Z_3+Z_1Z_4) \\ +q_4(Z_1Z_3+Z_2Z_4)+q_5(Z_2Z_3-Z_1Z_4)+q_6Z_3Z_4 \\ +q_7(Z_3^2-Z_4^2)+q_8(Z_3^2+Z_4^2)=0, \end{aligned} \quad (30)$$

where $q_i(i=1,2,\cdot,8)$ are the homogeneous coefficients of the quartic manifold surface in image space. Thus, the coefficients for G-manifold defined for each pose can be calculated using

$$A_{i1}=Z_{i1}^2+Z_{i2}^2 \quad (31)$$
$$A_{i2}=Z_{i1}Z_{i3}-Z_{i2}Z_{i4} \quad (32)$$
$$A_{i3}=Z_{i2}Z_{i3}+Z_{i1}Z_{i4} \quad (33)$$
$$A_{i4}=Z_{i1}Z_{i3}+Z_{i2}Z_{i4} \quad (34)$$
$$A_{i5}=Z_{i2}Z_{i3}-Z_{i1}Z_{i4} \quad (35)$$
$$A_{i6}=Z_{i3}Z_{i4} \quad (36)$$
$$A_{i7}=Z_{i3}^2-Z_{i4}^2 \quad (37)$$
$$A_{i8}=Z_{i3}^2+Z_{i4}^2 \quad (38)$$

where i is the pose index ranging from $i=(1,2,\cdots,n)$. Consolidating all the G-manifold equations results in the following over-constrained homogenous linear system on equation

$$Aq=0 \quad (39)$$

where

$$A=\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & \cdots & \cdots & \cdots & A_{18} \\ A_{21} & A_{22} & A_{23} & A_{24} & \cdots & \cdots & \cdots & A_{28} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & A_{n3} & A_{n4} & \cdots & \cdots & \cdots & A_{n8}, \end{bmatrix} \quad (40)$$

$$q=\begin{bmatrix} q_1 & q_2 & \cdots & q_8 \end{bmatrix}^T \quad (41)$$

The least square solution to this homogeneous system of equation can easily be found out using the full singular value decomposition of G-manifold coefficient matrix A. The last column of the right singular vector, which corresponds to the smallest singular value, is the least square solution. However, to generate a mechanism, at least two dyads are required. Thus, the right singular vector corresponding to the second-most and third-most smallest singular value is also taken as a possible solution. Space spanned by these three orthonormal vectors reflect a family of possible dyads which can be used for given poses.

For dyads to make physical sense, following extra C-manifold constraints are required to be satisfied

$$q_1q_6+q_2q_5-q_3q_4=0 \quad (42)$$
$$2q_1q_7-q_2q_4-q_3q_5=0 \quad (43)$$

An analytical solution to the minimization problem to satisfy the constraints gives a quartic equation. As a result, up to four unique dyads are generated. Combining any of the two dyads results in a solution four-bar mechanism.

The above methodology for motion fitting works for five or more poses when the system of equation is fully-constrained or over-constrained. To handle three or four pose under-constrained cases, additional Motion Synthesis constraints (MOC) also called geometric constraints outlined in [8] are used to specify the position of fixed or moving pivots using line or point constraints.

As a result, the path synthesis problem is solved and prospective solutions are generated. Using this motion synthesis algorithm also enables us to simultaneously carry out type and dimensional synthesis. However, solutions generated might be characterized by circuit and branch defects i.e. reaching all the path points in a given assembly mode might be impossible.

5 Mixed synthesis algorithm

A key advantage of the methodology outlined above is that it can handle both motion and mixed synthesis problems within it seamlessly. Various permutations of $(0,1,\cdots,m)$ poses and $(0,1,\cdots,n)$ path point problems are presented in Table 1. The legends in the table are MOC = Motion Synthesis constraint [8], MIC = Mixed Synthesis Constraint, FD = Fully Defined, and X=trivial or undefined. The * refers to conditions where a Fourier task curve with just four points needs to be fitted and would have unsymmetrical descriptors.

Motion Synthesis constraints can be used to specify the position of fixed or moving pivots using line or point constraints or any other compatible geometric constraint; see [8] for details. Mixed Synthesis Constraints, described earlier, involve constraints on actuating pivot, coupler dimensions, and other mechanism parameters. Fully Defined entails that no extra constraints are needed to exactly or least square solve the mixed synthesis Eq. (17). If any of the MSP fitting problem or Motion fitting problem is under-constrained, the mixed synthesis problem is under-constrained.

Except for the cases involving zero poses, Eq. (17) can be used to solve the generalized problem. When the synthesis problem has zero poses, the Fourier descriptor based path synthesis algorithm [14] is used. It must be noted that except for the case

TABLE 1: Various Possibilities for Unified Motion, Path and Mixed Synthesis Problem

		Path Points					
		0	1	2	3	4	n
Poses	0	X	X	X	X	FD*	FD
	1	X	X	X	2 MIC*	2 MIC	2 MIC
	2	X	X	1 MIC*	1 MIC	1 MIC	1 MIC
	3	2 MOC	1 MOC*	FD	FD	FD	FD
	4	1 MOC	FD	FD	FD	FD	FD
	5	FD	FD	FD	FD	FD	FD
	m	FD	FD	FD	FD	FD	FD

where there are no poses, the synthesis calculates both type and dimensions. However, cases with only path points are branch defect free.

The algorithm to solve the Alt-Burmester problems can be summarized as follows

Algorithm 1: Algorithm for Unified Motion, Path and Mixed Synthesis

Input: Path points and Poses
if *n(Path points+poses)≥ 4* **then**
  Calculate task curve fourier descriptors
continue
if *n(Pose)=0* **then**
  perform Fourier based path synthesis
else
  perform mixed synthesis
end
Output: Synthesized mechanism

6 Examples

In this section, some examples are discussed to illustrate the usefulness of proposed algorithm. First example aims to validate the approach by using path and poses from an existing mechanism. Second example solves a fully constrained mixed synthesis problem involving three poses and ten path points. Third example tests our algorithm for an over-constrained mixed synthesis problem with four poses and five path points. Fourth and fifth examples deal with under-constraint mixed synthesis cases and require additional mixed- and motion-constraints, respectively. Demonstrating valid results from each of these cases proves the robustness of proposed algorithm an all possible situations. It also displays the flexibility of the algorithm and its ability to incorporate various constraints.

6.1 Example 1: Existing Mechanism

To validate the proposed mixed synthesis algorithm, points and poses from a known planar four-bar mechanism are taken and then our algorithm is used to synthesize mechanisms. Ideally, we should get the exact same mechanism. However, a similar mechanism is also acceptable since approximations can occur at various steps, from task curve generation to algebraic fitting of the pose data. Max angular deviation for poses is within 1° for displayed configuration.

A sample mechanism displayed in Fig. 3 is used to generate seven path and three pose constraints. The mechanism has been defined using the position of its fixed pivots, moving pivots and coupler coordinates as shown in Table 2.

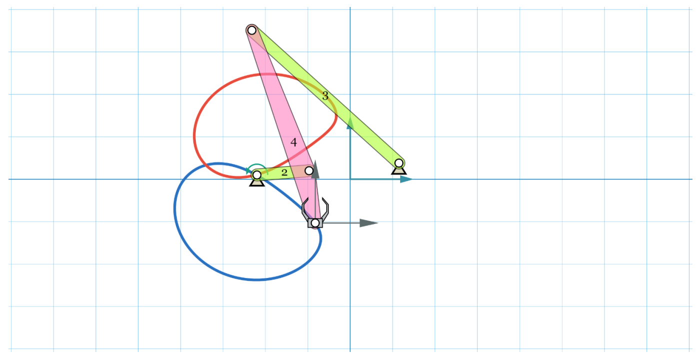

FIGURE 3: Example 1: Known target mechanism

TABLE 2: Example 1: Sample mechanism design parameters as shown in Fig. 3

Point	X	Y
Input link fixed pivot	−2.20	0.10
Input link moving pivot	−0.97	0.20
Output link fixed pivot	1.15	0.38
Output link moving pivot	−2.32	3.51
Coupler point	−0.825	−1.033

The sampled poses and path points are listed in Table 3. These constraints are used as input to mixed synthesis algorithm. Four solution dyads are output as listed in Table 4. This also allows us to reverse-engineer a known mechanism since there are six planar four-bars that can satisfy the given constraints. The closest dyads to original mechanism have been visualized in Fig 4. It is observed that the mechanism generated is very similar to the original mechanism. This approximate result is due to the best-fitted low harmonic task curve following the original coupler curve closely but not exactly.

TABLE 3: Example 1: Input constraint data

S.No	Constraint	X	Y	ζ
1	Pose	−0.825	−1.033	1.604
2	Pose	−1.655	−0.272	1.169
3	Point	−2.640	0.290	
4	Point	−3.500	0.310	
5	Point	−4.050	−0.210	
6	Pose	−4.102	−1.052	0.923
7	Point	−3.580	−1.870	
8	Point	−2.630	−2.340	
9	Point	−1.560	−2.270	
10	Point	−0.800	−1.740	

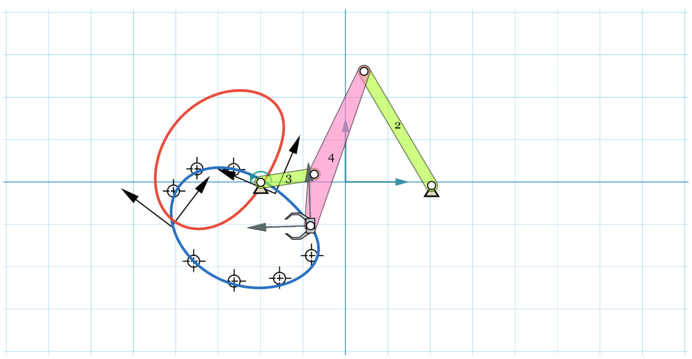

FIGURE 4: Example 1: Mechanism generated using mixed synthesis algorithm

TABLE 4: Example 1: Output dyad data

Dyad	Fixed Pivot	Moving Pivot	Coupler Point
1	0.679, 4.534	0.046, 5.419	−0.825, −1.033
2	−6.845, 8.102	−2.147, 3.587	−0.825, −1.033
3	2.031, −0.084	0.438, 2.611	−0.825, −1.033
4	−1.998, −0.009	−0.737, 0.182	−0.825, −1.033

6.2 Example 2: Fully constrained mixed synthesis

In this example, the input constraint data consists of three poses and ten path points which fully constrains the mixed synthesis problem. The constraint data input to mixed synthesis algorithm has been displayed in Table 5. The two dyads generated as output have been shown in Table 6. The final mechanism has been displayed in Fig 5 and Fig 6. It can be observed that a good match has been established with the constraints. Max angular deviation at nearest points to specified poses is within 1°. Note that in this case, the path-orientation relationship has an exact solution i.e. MSP are uniquely determined using SVD.

TABLE 5: Example 2: Input constraint data

S.No	Constraint	X	Y	ζ
1	Pose	−9.853	1.139	0.336
2	Point	−7.700	2.020	
3	Point	−5.880	2.060	
4	Point	−2.800	1.680	
5	Point	−1.540	0.690	
6	Point	−0.370	−0.020	
7	Pose	0.754	−0.400	6.150
8	Point	2.560	−0.770	
9	Point	3.660	−0.720	
10	Point	4.680	−0.570	
11	Point	5.850	−0.110	
12	Point	6.790	0.570	
13	Pose	7.482	1.309	0.530

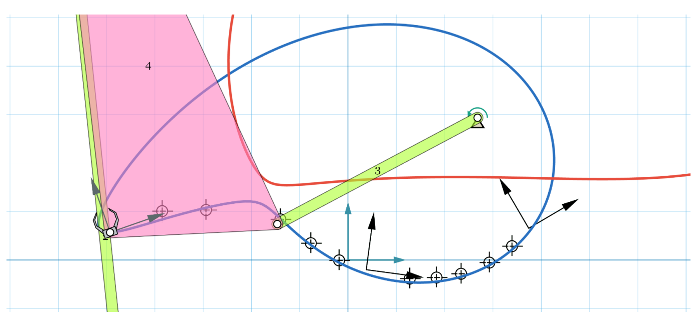

FIGURE 5: Example 2: Fully-constrained mixed synthesis for three poses and ten path points

One of the major advantages of mixed synthesis is the additional flexibility it imparts to users while specifying inputs and

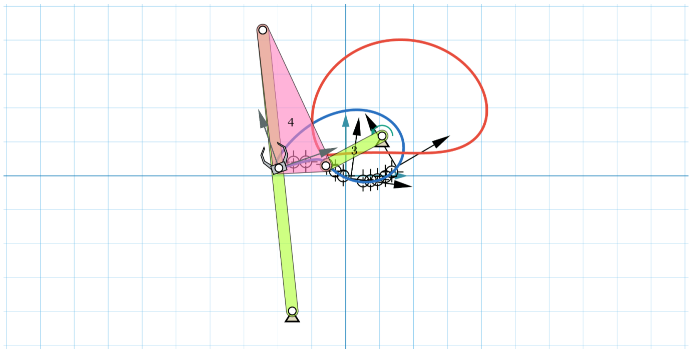

FIGURE 6: Example 2: Mixed synthesis- zoomed out

TABLE 6: Example 2: Output dyad data

Dyad	Fixed Pivot	Moving Pivot	Coupler Point
1	$-7.876, -19.931$	$-12.228, 21.484$	$-9.853, 1.139$
2	$5.363, 5.873$	$-2.929, 1.483$	$-9.853, 1.139$

generating good solutions. Using a pure motion synthesis algorithm, the user would have to input all the data as poses even if the problem demanded otherwise. This would lead to an over constrained motion problem, which when solved using existing kinematic mapping based algebraic fitting approach [7, 8, 9, 10] usually produced poor solutions. A comparable motion synthesis problem for the same path points is displayed in Fig 7. It can be observed that the solution provides a very poor fit to the given constraints.

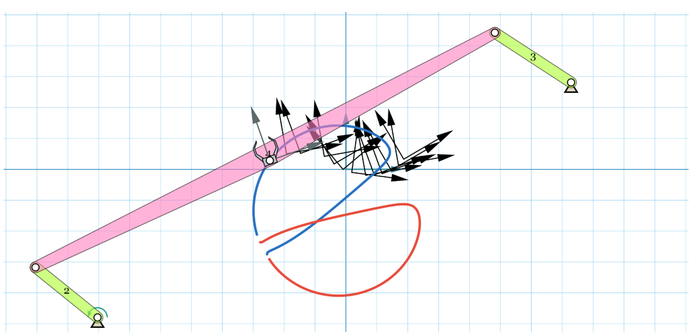

FIGURE 7: Example 2: Over-constrained motion synthesis for thirteen poses produces a poor solution.

6.3 Example 3: Over-constrained mixed synthesis

In this example, the input constraint data consists of four poses and five path points. Greater than three input poses over constrains the MSP calculation. This make mixed synthesis problem over-constrained. The constraint data input to mixed synthesis algorithm is shown in Table 7. The four dyads generated as output have been shown in Table 8. One of the final mechanisms has been displayed in Fig 8. The mechanism is satisfactory as it closely follows the path input specified. Max angular deviation at nearest points to specified poses is within 15° for the displayed dyads. Note that in this case, the path-orientation relationship has an approximate solution i.e. least square fitted MSP are determined using SVD.

TABLE 7: Example 3: Input constraint data

S.No	Constraint	X	Y	ζ
1	Pose	-2.750	-1.947	0.000
2	Pose	-0.674	-1.947	0.668
3	Point	-0.290	-0.180	
4	Point	-0.580	1.070	
5	Pose	-1.662	1.797	0.887
6	Point	-3.100	2.090	
7	Point	-3.960	2.200	
8	Point	-5.250	2.260	
9	Pose	-5.985	2.196	2.142

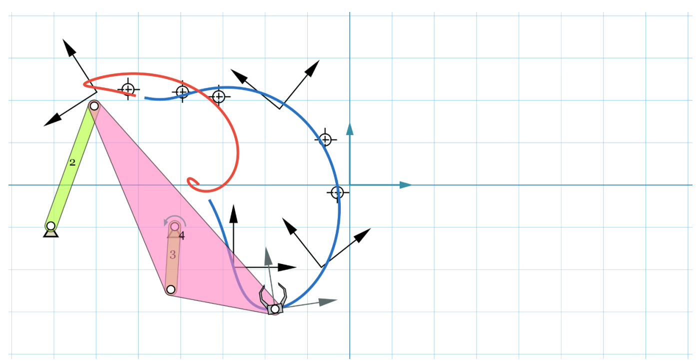

FIGURE 8: Example 3: Over-constrained mixed synthesis for four poses and five path points

6.4 Example 4: Under-constrained mixed synthesis with mixed constraints

This example deals with the under-constrained mixed synthesis problem where two poses and four path points are specified in the input. Lesser than three input poses makes MSP fitting

TABLE 8: Example 3: Output dyad data

Dyad	Fixed Pivot	Moving Pivot	Coupler Point
1	$-5.603, -0.821$	$-6.769, -1.098$	$-2.750, -1.947$
2	$-7.071, -0.967$	$-7.871, 1.940$	$-2.750, -1.947$
3	$-7.610, 1.820$	$-12.065, -16.791$	$-2.750, -1.947$
4	$-4.140, -0.984$	$-5.258, -1.968$	$-2.750, -1.947$

under-constrained. This makes mixed synthesis problem under-constrained. To solve for MSP, an additional mixed constraint is required which could specify any of z_1, z_2, z_3. The constraint data input to mixed synthesis algorithm has been displayed in Table 9. A MIC is used to specify z_2 by specifying preferred location of a fixed joint at point $(1,2)$. The four dyads generated as output have been shown in Table 10. One of the final mechanisms has been displayed in Fig 9. It can be observed that the generated mechanism closely satisfies path and mixed constraints and thus is useful. Max angular deviation at nearest points to specified poses is within 45° in displayed dyads. Note that in this case, the path-orientation relationship has infinite solutions and the use of MIC restricts the solution space to a unique MSP.

TABLE 9: Example 4: Input constraint data

S.No	Constraint	X	Y	ζ
1	Pose	4.962	-0.514	0.134
2	Point	3.850	-1.480	
3	Point	1.920	-0.740	
4	Point	0.850	0.760	
5	Point	3.360	1.650	
6	Pose	4.900	1.178	0.510

TABLE 10: Example 4: Output dyad data

Dyad	Fixed Pivot	Moving Pivot	Coupler Point
1	$14.769, 24.102$	$-2.837, -4.531$	$4.962, -0.514$
2	$-12.641, -7.054$	$-3.133, -0.666$	$4.962, -0.514$
3	$1.261, 2.610$	$1.739, -1.431$	$4.962, -0.514$
4	$4.357, 0.283$	$6.270, -0.729$	$4.962, -0.514$

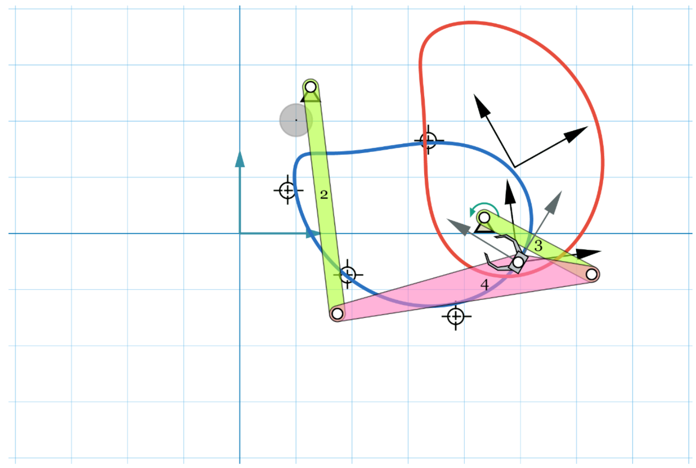

FIGURE 9: Example 4: Under-constrained mixed synthesis for two poses and four path points using additional mixed constraint

6.5 Example 5: Under-constrained mixed synthesis with motion constraints

This example deals with another under-constrained mixed synthesis problem where three poses and one path point is specified in input. This case is under-constraint because the total pose and path constraints are just four. Even though three poses specified can be used to calculate the MSP, an additional motion constraint is required to solve the synthesis problem. The constraint data input to mixed synthesis algorithm has been displayed in Table 11. A line constraint is used as MOC in the example presented. The line segment is defined by its end points $(-4,1)$ and $(1,4)$. The three dyads generated as output have been shown in Table 12. One of the final mechanisms has been displayed in Fig 10. It can be observed that the generated mechanism closely satisfies path constraints. Max angular deviation at nearest points to specified poses is within 1° for the displayed dyads. Also, both the fixed pivots fall on the line constraint specified. Thus, the synthesis problem is successfully solved. Note that in this case, it is not the path-orientation relationship that is under-defined but the algebraic fitting algorithm which requires at-least five poses to be fully defined.

TABLE 11: Example 5: Input constraint data

S.No	Constraint	X	Y	ζ
1	Pose	-2.018	-1.391	0.146
2	Pose	0.288	-1.115	0.287
3	Pose	2.895	1.253	1.487
4	Point	1.490	0.080	

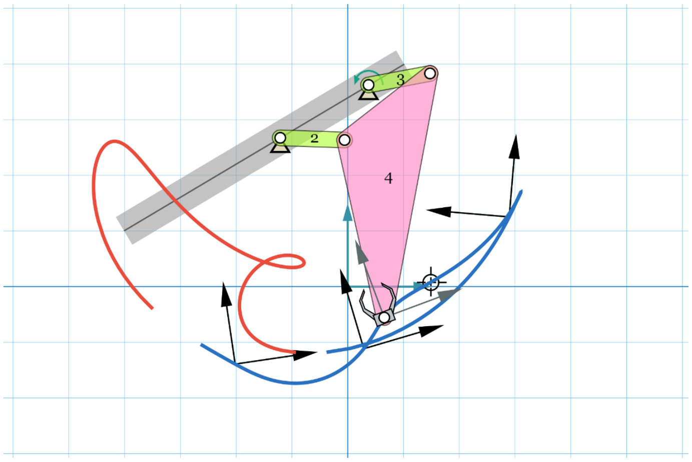

FIGURE 10: Example 5: Under-constrained mixed synthesis for three poses and one path points using additional motion constraint

TABLE 12: Example 5: Output dyad data

Dyad	Fixed Pivot	Moving Pivot	Coupler Point
1	$0.374, 3.624$	$-0.296, 2.725$	$-2.018, -1.391$
2	$-3.224, 1.466$	$-4.027, 3.045$	$-2.018, -1.391$
3	$-1.209, 2.674$	$-2.044, 1.878$	$-2.018, -1.391$

7 Conclusion

In this paper, we have presented a generalized m pose, n path mixed synthesis approach for four-bar mechanisms. Original contributions of this paper include the closed form relationship between coupler orientation and coupler path. Using this analytic relationship as the core, a novel framework is proposed to solve the mixed synthesis problem. Another novel feature is the use of task-driven motion synthesis algorithm within the framework to keep the computation cost at minimum and do simultaneous type and dimensional synthesis. Many examples have been presented to display the usefulness of the model. This approach is a step towards building a generalized synthesize algorithm which unifies path, function and motion synthesis.

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