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UNIFIED MOTION SYNTHESIS OF SPATIAL SEVEN-BAR PLATFORM MECHANISMS AND PLANAR-FOUR BAR MECHANISMS

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ABSTRACT

A unified motion generation algorithm that combines Spatial and Planar mechanism synthesis has been a hard problem in kinematics. In this paper, we present a new method to generate planar RRRR mechanisms and spatial 5-SS mechanisms using a unified algorithm. For a generalized spatial pose problem where all the poses fall on a plane, we show that there exist 1-∞ plane constraint solutions and 3-∞ planar-spherical solution dyads. We also show that for a spatial five pose problem where all poses lie on a plane, there exists a 2-∞ solution space of spherical and planar constraints. This multiplicity of solutions are intelligently constrained to find up to four circle constraints representing planar four-bar mechanism. Finally, examples are presented testing the proposed algorithm and verified using results from past publications.

1. INTRODUCTION

The Motion synthesis problem is the determination of the type and dimension of a kinematic mechanism to guide one of its links through several specified poses [1, 2]. This problem has a rich literature dealing with synthesis of planar, spherical, and spatial mechanisms. Burmester first solved the planar four-bar motion synthesis problem for five poses [3]. Since then, many researchers have studied the planar four-bar motion synthesis problem [4, 5, 6, 7, 8, 9, 10]. Ge et al. proposed an algebraic fitting based unified type and dimensional synthesis for planar mechanisms [11, 12].

Bodduluri and McCarthy [13] carry out finite position synthesis of spherical mechanisms by minimizing the normal distance in the image space. Lin [14] uses homotopy methods to generate spherical four-bar mechanisms for motion and path generation. Ruth and McCarthy [15] describe SphinxPC, a computer-aided design software system for spherical four-bar linkage synthesis. Brunnthaler et al. [16] use kinematic mapping to synthesize spherical four-bars. Zhuang et al. [17] have used an adaptive genetic algorithm to synthesize spherical four-bars. Li et al. [18] solve a more general *n*-discrete pose problem using kinematic mapping.

Innocenti [19] extended the notion of geometric constraints to the construction of 5-SS spatial platforms using spherical constraints and termed it the Spatial Burmester problem. It has been shown that a 5-SS mechanism can pass exactly through seven spatial poses. Liao and McCarthy [20] improved upon Innocenti's work and formulated a methodology for singularity analysis. Plecnik and McCarthy [21] used the 5-SS platform as a steering linkage. Li et al. [22] carry out both type and dimensional synthesis of platform linkages using an algebraic fitting approach to unify the spherical and planar constraints. Ge et al. [23, 24] have improved upon the algebraic fitting approach of Li et al. and incorporated constraints on pivot locations.

In the above classic approaches, planar, spherical, and spatial mechanism synthesis are considered as separate design prob-

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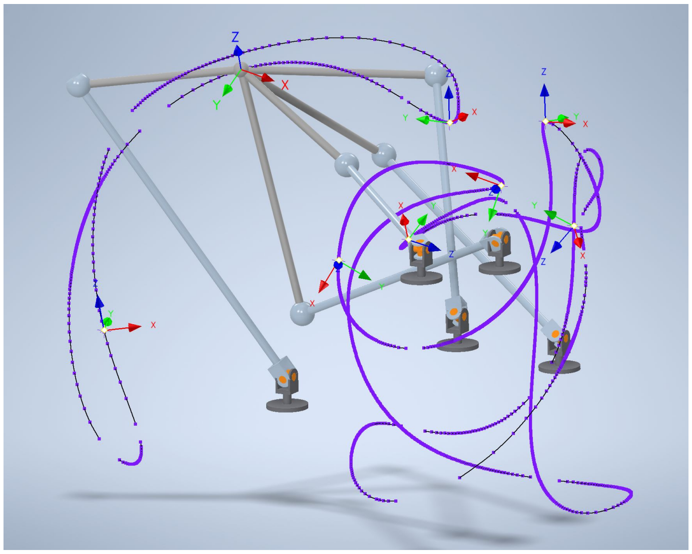


FIGURE 1: An example 5-SS mechanism

lems. An ideal approach would be one which can synthesize all types of mechanism using a unified framework. This serves as the motivation for this paper. Ge et al. [25] have proposed an approach that unifies synthesis of planar and spherical dyads. Ge et al. [26] also proposed a methodology to synthesize spatial RR dyads using an intersection of SS dyads. However, it requires at least seven poses to generate RR dyads which is not in agreement with the planar Burmester theory which says five poses are sufficient to generate unique RR dyads.

In this paper, we present a novel algebraic fitting based algorithm to unify planar and spatial mechanism synthesis by careful analysis of solution space dimensionality. We use a dual quaternion representation to describe both spatial and planar pose data. An example spatial 5-SS mechanism and a spatial RRRR mechanism are shown in Fig. 1 and Fig. 2a, respectively. First, the existing methodology, with minor tweaks, for spatial motion synthesis is presented. Since the focus of this paper is unification of spatial and planar synthesis, we explore the nature of solution space when all spatial poses lie on a plane. We show that a planar seven pose motion generation problem can lead at least 3-∞ Planar-Spherical dyad solutions due to rank reduction of bilinear constraint matrix. Using this knowledge, we tackle the spatial five-pose problem to find if a circle constraint exists. It is shown that at least 2-∞ planar or spherical solutions exist for a planar four bar mechanisms in space. In Fig. 2, we can see that moving the fixed pivot or the moving pivot perpendicular to the plane doesn't affect the coupler motion. Constraints are systematically placed on the solution space to extract the simplest RR dyad solutions.

The intellectual contributions of this paper can be summarized as 1) Unification of five pose planar Burmester problem and seven pose spatial Burmester problem into a single framework; 2) Existence of at least 3-∞ solutions when spatial poses lie on a plane and 3) Existence of at least 2-∞ solutions for planar four bars in space.

The remaining paper is organized as follows. Section 2 discusses the representation of spatial poses as dual quaternions. Section 3 reviews the unified constraint equation for plane and sphere constraints. Section 4 demonstrates the existing algorithm to carry out motion synthesis using seven spatial pose. Section 5 discusses the nature of solution space when all poses lie on a plane. Section 6 presents a systematic methodology to isolate RR dyads from the solution space representing SS and PS dyads. Section 7 illustrates some examples and Section 8 concludes the paper.

2. SPATIAL DISPLACEMENT REPRESENTATION

In this section, we review the relations used to express spatial Poses in the form of dual quaternions [27, 28]. Let a moving rigid body in space be denoted by a coordinate frame M attached to it. A point on moving body with respect to M can be represented using homogeneous coordinates $\mathbf{c} = (c_1, c_2, c_3, c_4)$. The same point is defined as $\mathbf{C} = (C_1, C_2, C_3, C_4)$ in fixed frame F. The point coordinate transformation from moving frame M to fixed frame F is given as

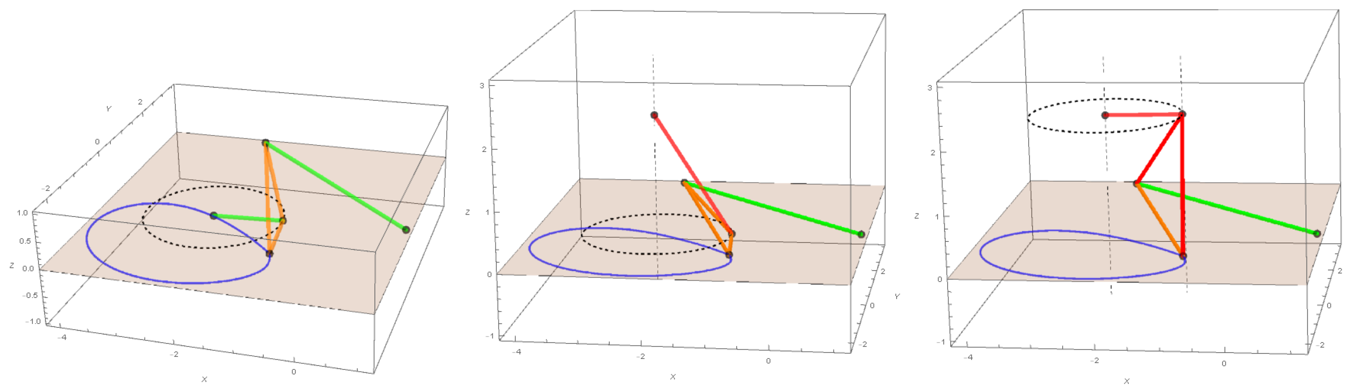
$$\mathbf{C} = \begin{bmatrix} \mathbf{R} & \mathbf{d} \\ 0 & 1 \end{bmatrix} \mathbf{c} \tag{1}$$

where $\mathbf{R}$ is the rotation matrix that describes the orientation of M relative to F and $\mathbf{d} = (d_1, d_2, d_3)$ is the vector from origin of F to M. The pose orientation can be described using the Euler-Rodrigues parameters which involves rotation axis and angle. The axis is represented by a unit vector $\mathbf{s} = (s_x, s_y, s_z)$ and rotation angle θ.

A spatial pose can be represented using a unit dual quaternion $\mathbf{Q} = (\mathbf{q}, \mathbf{g})$ where the real part is quaternion $\mathbf{q} = (q_1, q_2, q_3, q_4)$ and dual part is quaternion $\mathbf{g} = (g_1, g_2, g_3, g_4)$. It satisfies the relations

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1 \tag{2}$$

$$q_1 g_1 + q_2 g_2 + q_3 g_3 + q_4 g_4 = 0 \tag{3}$$



(a) Example Spatial RRRR Four-Bar Mechanism (b) Translating Fixed Pivot (c) Translating Moving Pivot

FIGURE 2: Translating the fixed pivot or the moving pivot perpendicular to plane does not change the coupler motion resulting in 2-∞ solutions

The dual quaternion **Q** can be calculated as follows

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} s_x sin(\theta/2) \\ s_y sin(\theta/2) \\ s_z sin(\theta/2) \\ cos(\theta/2) \end{bmatrix} \quad (4)$$

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -d_3 & d_2 & d_1 \\ d_3 & 0 & -d_1 & d_2 \\ -d_2 & d_1 & 0 & d_3 \\ -d_1 & -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \quad (5)$$

The relationships between dual quaternion **Q** and the rotation matrix **R** and displacement vector **d** can be given as

$$\mathbf{R} = \begin{bmatrix} q_4^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_4q_3) & 2(q_1q_3+q_4q_2) \\ 2(q_1q_2+q_4q_3) & q_4^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_4q_1) \\ 2(q_1q_3-q_4q_2) & 2(q_2q_3+q_4q_1) & q_4^2-q_1^2-q_2^2+q_3^2 \end{bmatrix} \quad (6)$$

$$\mathbf{d} = -2 \begin{bmatrix} g_4q_1-g_1q_4+g_2q_3-g_3q_2 \\ g_4q_2-g_2q_4+g_3q_1-g_1q_3 \\ g_4q_3-g_3q_4+g_1q_2-g_2q_1 \end{bmatrix} \quad (7)$$

The use of dual quaternion **Q** leads to a compact constraint equation for SS dyads as can be seen in the next section.

3. UNIFIED REPRESENTATION OF SPHERICAL AND PLANAR CONSTRAINTS

In this section, we review the unified constraint relation used by previous works on spatial mechanism synthesis [24] in their work. Some of the equations have been modified for clarity. For a spatial Spherical-Spherical (SS) dyad, its coupler point $C = (C_1, C_2, C_3, C_4)$ is geometrically constrained on a sphere whose radius and center are represented by homogeneous coordinates $A = (A_0, A_1, A_2, A_3, A_4)$. This constraint can be given as

$$2A_1C_1 + 2A_2C_2 + 2A_3C_3 + A_0C_4 = A_4\left(\frac{C_1^2+C_2^2+C_3^2}{C_4}\right), \quad (8)$$

$$A_4^2r^2 - A_0A_4 = A_1^2 + A_2^2 + A_3^2, \quad (9)$$

where r is the radius of sphere formed by SS dyad. When $A_4 = 0$, the geometric constraint equation represents a plane described by a Planar-Spherical dyad. Eq (8) consists of seven independent dyadic parameters since A_i and C_i are homogeneous in nature.

The final dyadic constraint equation is obtained by substituting the fixed frame coupler point (**C**) in geometric constraint equation Eq (8) with moving coordinate (**c**) according to transformation relationships in Eq (1). On collecting similar terms, we can restructure constraint equation as follows

$$\sum_{i=0}^{16} K_i P_i = 0 \quad (10)$$

where constraint space coefficients (CSCs) $(K_0, K_1, \ldots K_{16})$ are given as

$$
\begin{aligned}
K_0 &= 1,\\
K_1 &= 2(q_1^2 - q_2^2 - q_3^2 + q_4^2),\\
K_2 &= 4(q_1q_2 + q_3q_4),\\
K_3 &= 4(q_1q_3 - q_2q_4),\\
K_4 &= 4(g_4q_1 + g_3q_2 - g_2q_3 - g_1q_4),\\
K_5 &= 4(q_1q_2 - q_3q_4),\\
K_6 &= 2(-q_1^2 + q_2^2 - q_3^2 + q_4^2),\\
K_7 &= 4(q_2q_3 + q_1q_4),\\
K_8 &= 4(-g_3q_1 + g_4q_2 + g_1q_3 - g_2q_4),\\
K_9 &= 4(q_1q_3 + q_2q_4),\\
K_{10} &= 4(q_2q_3 - q_1q_4),\\
K_{11} &= 2(-q_1^2 - q_2^2 + q_3^2 + q_4^2),\\
K_{12} &= 4(g_2q_1 - g_1q_2 + g_4q_3 - g_3q_4),\\
K_{13} &= 4(-g_4q_1 + g_3q_2 - g_2q_3 + g_1q_4),\\
K_{14} &= 4(-g_3q_1 - g_4q_2 + g_1q_3 + g_2q_4),\\
K_{15} &= 4(g_2q_1 - g_1q_2 - g_4q_3 + g_3q_4),\\
K_{16} &= 4(-g_1^2 - g_2^2 - g_3^2 - g_4^2),
\end{aligned}
\tag{11}
$$

and the constraint space parameters (CSPs) $(P_0, P_1, \ldots P_{16})$ are given as

$$
\begin{gathered}
P_0 = A_0c_4 - A_4c_4\left(\frac{c_1^2}{c_4^2} + \frac{c_2^2}{c_4^2} + \frac{c_3^2}{c_4^2}\right),\\
P_1 = A_1c_1, \quad P_2 = A_2c_1, \quad P_3 = A_3c_1, \quad P_4 = A_4c_1,\\
P_5 = A_1c_2, \quad P_6 = A_2c_2, \quad P_7 = A_3c_2, \quad P_8 = A_4c_2,\\
P_9 = A_1c_3, \quad P_{10} = A_2c_3, \quad P_{11} = A_3c_3, \quad P_{12} = A_4c_3,\\
P_{13} = A_1c_4, \quad P_{14} = A_2c_4, \quad P_{15} = A_3c_4, \quad P_{16} = A_4c_4
\end{gathered}
\tag{12}
$$

Eq (10) is referred as the Constraint equation (C-equation) and it consists of 17 homogeneous CSPs P_i. A_i and c_j are referred as the Dyadic parameters. The CSPs are subjected to nine additional bi-linear constraints as follows

$$
\begin{aligned}
\frac{P_1}{P_{13}} = \frac{P_2}{P_{14}} = \frac{P_3}{P_{15}} = \frac{P_4}{P_{16}} = \frac{c1}{c4} = \lambda_1\\
\frac{P_5}{P_{13}} = \frac{P_6}{P_{14}} = \frac{P_7}{P_{15}} = \frac{P_8}{P_{16}} = \frac{c2}{c4} = \lambda_2\\
\frac{P_9}{P_{13}} = \frac{P_{10}}{P_{14}} = \frac{P_{11}}{P_{15}} = \frac{P_{12}}{P_{16}} = \frac{c3}{c4} = \lambda_3
\end{aligned}
\tag{13}
$$

Thus, the dimension of solution space for CSPs P_i in Eq (10) is seven. This is in accordance with the seven-pose spatial Burmester theory to generate unique solution.

These Spherical and Planar constraints can also represent other spatial linkages. The geometric constraint for a Universal-Spherical (TS) link and an RRS open chain with intersecting axes for the RR joints is a sphere. The constraint for a RRS with parallel axis, a Prismatic-Revolute-Spherical and a Revolute-Prismatic-Spherical open chain is a plane.

Next, we discuss the algorithm which uses Eq (10) and Eq (13) to synthesize platform mechanisms constrained by spherical or planar geometric constraints.

4. MOTION SYNTHESIS FOR SEVEN OR MORE GENERAL SPATIAL POSES

This section, adapts the work done by Ge et al. [25, 26, 24] with minor modifications. The motion synthesis algorithm takes a set of spatial poses as input. To get unique solutions, at least seven spatial poses are required by the algorithm. For less than seven poses, an infinite number of spherical and planar constraints exist. The computation is carried out in three distinct steps and the result is a set of unique dyads defined by their homogeneous coordinates A_i and c_i.

First, the pose constraints are enforced using Eq (10). Then, the bi-linear constraints are imposed using Eq (13) to get unique values of the CSPs P_i. Lastly, these values are mapped to their respective dyadic representation using A_i and c_i.

4.1 Applying Pose Constraints

First, the CSCs (K_i) for each spatial pose are calculated using Eq (10). For an n-pose problem, these values can be consolidated into a system of equations as follows

$$
\begin{bmatrix} K_{1,0} & K_{1,1} & \cdots & K_{1,16}\\ K_{2,0} & K_{2,1} & \cdots & K_{2,16}\\ \vdots & \vdots & \ddots & \vdots\\ K_{n,0} & K_{i,1} & \cdots & K_{n,16} \end{bmatrix}
\begin{bmatrix} P_0\\ P_1\\ \vdots\\ P_{16} \end{bmatrix} = [\mathbf{K}]\mathbf{P} = 0
\tag{14}
$$

where $[\mathbf{K}]$ represents the coefficient matrix. The null-space of this system of equation can be analyzed using Singular Value Decomposition (SVD); i.e., $[\mathbf{K}]$ is factorized into

$$
[\mathbf{K}] = [\mathbf{U}][\mathbf{W}][\mathbf{V}]^T
\tag{15}
$$

where $[\mathbf{U}]$ is an $n \times n$ orthonormal matrix whose columns represent the left singular vectors of $[\mathbf{K}]$; $[\mathbf{W}]$ is the $n \times 17$ diagonal matrix whose elements are square root of eigenvalues of $[\mathbf{K}][\mathbf{K}]^T$; and $[\mathbf{V}]^T$ is the 17×17 orthonormal matrix whose columns represent the right singular vectors. The solution space is the vector space spanned by the right singular vectors corresponding to singular values having negligible magnitude.

Since it is well known that a spatial SS dyad can exactly pass through a maximum of seven poses, ten right singular vectors corresponding to singular values having the least magnitude are selected. Basically, we apply seven constraints on a 17-dimensional solution space to get a ten-dimensional solution subspace. When more than seven poses are inputted, the span of these ten singular vectors represent the solution space in the least square sense. Thus, the solution vector $\mathbf{P}$ can be given as a linear combination of ten singular vectors $\mathbf{v_i}$ i.e.

$$\mathbf{P} = \sum_{i=1}^{10} \alpha_i \mathbf{v_i} = \begin{bmatrix}\mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_{10}}\end{bmatrix} \begin{bmatrix}\alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{10}\end{bmatrix} = [\mathbf{V}]\alpha. \tag{16}$$

4.2 Applying Bi-linear Constraints

Next, we apply the nine bi-linear constraints on the solution space as described in Eq (13). To enforce the bi-linear constraints, $[\mathbf{V}]$ in Eq (16) is broken down into smaller system of equations as follows

$$\begin{bmatrix}P_1 \\ P_2 \\ P_3 \\ P_4\end{bmatrix} = \begin{bmatrix}v_{1,1} & \cdots & v_{10,1} \\ v_{1,2} & \cdots & v_{10,2} \\ v_{1,3} & \cdots & v_{10,3} \\ v_{1,4} & \cdots & v_{10,4}\end{bmatrix} \begin{bmatrix}\alpha_1 \\ \vdots \\ \alpha_{10}\end{bmatrix} = W_1\alpha \tag{17}$$

$$\begin{bmatrix}P_5 \\ P_6 \\ P_7 \\ P_8\end{bmatrix} = \begin{bmatrix}v_{1,5} & \cdots & v_{10,5} \\ v_{1,6} & \cdots & v_{10,6} \\ v_{1,7} & \cdots & v_{10,7} \\ v_{1,8} & \cdots & v_{10,8}\end{bmatrix} \begin{bmatrix}\alpha_1 \\ \vdots \\ \alpha_{10}\end{bmatrix} = W_2\alpha \tag{18}$$

$$\begin{bmatrix}P_9 \\ P_{10} \\ P_{11} \\ P_{12}\end{bmatrix} = \begin{bmatrix}v_{1,9} & \cdots & v_{10,9} \\ v_{1,10} & \cdots & v_{10,10} \\ v_{1,11} & \cdots & v_{10,11} \\ v_{1,12} & \cdots & v_{10,12}\end{bmatrix} \begin{bmatrix}\alpha_1 \\ \vdots \\ \alpha_{10}\end{bmatrix} = W_3\alpha \tag{19}$$

$$\begin{bmatrix}P_{13} \\ P_{14} \\ P_{15} \\ P_{16}\end{bmatrix} = \begin{bmatrix}v_{1,13} & \cdots & v_{10,13} \\ v_{1,14} & \cdots & v_{10,14} \\ v_{1,15} & \cdots & v_{10,15} \\ v_{1,16} & \cdots & v_{10,16}\end{bmatrix} \begin{bmatrix}\alpha_1 \\ \vdots \\ \alpha_{10}\end{bmatrix} = W_4\alpha \tag{20}$$

The bi-linear constraints can now be imposed as follows

$$W_1\alpha = \lambda_1 W_4\alpha \tag{21}$$
$$W_2\alpha = \lambda_2 W_4\alpha \tag{22}$$
$$W_3\alpha = \lambda_3 W_4\alpha \tag{23}$$

This generalized eigenvalue problem can be written as

$$\begin{bmatrix}W_1 - \lambda_1 W_4 \\ W_2 - \lambda_2 W_4 \\ W_3 - \lambda_3 W_4\end{bmatrix}_{12\times 10} \alpha_{10\times 1} = [\mathbf{W}]\alpha = 0. \tag{24}$$

As we can note, Eq. (24) is a system of twelve equations with ten unknowns. For this system of equations to be compatible and have a non-trivial solution, the matrix $[\mathbf{W}]$ need to be rank reduced i.e. have a rank of nine. To accomplish this requirement, determinant of any three resulting 10×10 square matrices need to be zero. This results in three non-linear polynomial equations in variables $\lambda_1, \lambda_2, \lambda_3$ which can be solved using polynomial homotopy based solvers (Bertini) or symbolic solvers (Mathematica) to get all possible solutions. This results in upto 20 solution pairs [19]. Once the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ are known, they can be substituted back in Eq. (24) and null space analysis can be done to find the solution vectors α. Finally, the solution CSPs P_i can be calculated using Eq. (16).

4.3 Finding Dyadic Parameters

Once the CSPs are known, the dyadic parameters finally can be calculated. For a Planar geometric constraint, $A_4 = 0$ due to which P_4, P_8, P_{12}, and $P_{16} = 0$, while for spherical geometric constraints, $A_4 \neq 0$, i.e. P_4, P_8, P_{12}, and P_{16} are nonzero.

If the constraint is spherical, its dyadic parameters represents the fixed pivot coordinates (A_i) and moving pivot coordinates (c_i) and are described using the following inverse relationships

$$A_0 : A_1 : A_2 : A_3 : A_4 = P_0 + \frac{P_4^2}{P_{16}} + \frac{P_8^2}{P_{16}} + \frac{P_{12}^2}{P_{16}} : P_{13} : P_{14} : P_{15} : P_{16} \tag{25}$$

$$c_1 : c_2 : c_3 : c_4 = P_4 : P_8 : P_{12} : P_{16} \tag{26}$$

Also, the radius of sphere can be calculated as

$$r = \sqrt{\frac{P_{13}^2}{P_{16}^2} + \frac{P_{14}^2}{P_{16}^2} + \frac{P_{15}^2}{P_{16}^2} + \frac{P_4^2}{P_{16}^2} + \frac{P_8^2}{P_{16}^2} + \frac{P_{12}^2}{P_{16}^2} + \frac{P_0}{P_{16}}} \tag{27}$$

If the constraint is planar, its dyadic parameters represents the plane normal vector (A_i) and moving pivot coordinates (c_i)

$$\begin{aligned} A_0 : A_1 : A_2 : A_3 &= P_0/2 : P_1 : P_2 : P_3 \\ &= P_0/2 : P_5 : P_6 : P_7 \\ &= P_0/2 : P_9 : P_{10} : P_{11} \\ &= P_0/2 : P_{13} : P_{14} : P_{15} \end{aligned} \tag{28}$$

$$\begin{aligned} c_1 : c_2 : c_3 : c_4 &= P_1 : P_5 : P_9 : P_{13} \\ &= P_2 : P_6 : P_{10} : P_{14} \\ &= P_3 : P_7 : P_{11} : P_{15} \end{aligned} \tag{29}$$

Thus, we obtain up to 20 unique solution constraints which can satisfy the initial spatial pose problem. Picking any five of the available 20 can lead up to 15,504 unique solution one degree-of-freedom mechanisms.

5. MOTION SYNTHESIS FOR SEVEN OR MORE PLANAR SPATIAL POSES

In this section, we explore a special case of the spatial motion synthesis problem where all the poses lie on a plane. In this edge case, the algorithm outlined by Ge et al. [24] fails to generate unique solutions.

For a seven or more planar pose problem, we can easily impose the pose constraints using nullspace analysis on Eq (14). However, when applying the bi-linear constraints using Eq (24), we find that the symbolic matrix **W** ends up being of rank nine. As a result, the conditions of having zero magnitude determinants of 10×10 submatrix are trivially satisfied. Thus, any values of $\lambda_1, \lambda_2, \lambda_3$ can be selected to construct a 3-∞ number of plane constraints.

Since $\lambda_1 : \lambda_2 : \lambda_3 = c_1 : c_2 : c_3$ from Eq. (13), we find that these 3-∞ solutions can be generated by changing the coordinates of coupler point on moving frame in x,y and z directions.

To calculate this solution space, we sample i random values of $\lambda_1, \lambda_2, \lambda_3$ and calculate their corresponding solution vectors α using null space analysis of Eq. (24). To find the solution space spanned by these vectors α_i, we augment the vectors into a matrix $[\alpha_1, \alpha_2, \cdots, \alpha_i]$, calculate its SVD and take four leftmost vectors (u_1, u_2, u_3, u_4) of the left singular vectors which represent the basis of solution space. Back substituting these values in Eq. (16) gives us the solution space of CSPs P_i given as

$$\mathbf{P} = \begin{bmatrix} v_1 & v_2 & \cdots & v_{10} \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = [\mathbf{V}][\mathbf{U}]\beta. \tag{30}$$

where β represents four homogeneous free variables. Thus, we can analytically find the 3-∞ solutions to motion synthesis problem for planar poses in space. Next we solve the five pose planar Burmester problem in space. Since the poses lie on a plane, they can also be achieved using Planar-Revolute dyads, which are much simpler than Planar-Spherical dyads.

6. MOTION SYNTHESIS FOR FIVE PLANAR POSES

In this section, we explore an even more special case of the spatial motion synthesis problem where all the poses lie on a plane and dyadic constraints are circles instead of spheres or planes. It is well known that a planar four-bar mechanism can pass exactly through a maximum of five poses. Our approach in this section would be imposing apt constraints on the C-equation so that we can generate these four-bar RR dyads. Since the problem has two less pose constraints than seven poses, there exist at least 2-∞ SS or PS dyads. Due to this under-constrained nature of problem, the above discussed approaches fail to work and need to be upgraded.

First, the five or more pose constraints are applied using null-space analysis on Eq (14) and results in a 12-dimensional solution space in contrast to 10-dimensional as seen in Eq. (16). Thus, if the number of singular values with negligible magnitude are 12, we have the possibility of finding exact planar RR dyads. Otherwise, the 12-dimensional solution space represents a least-square solution.

$$\mathbf{P} = \sum_{i=1}^{12} \alpha_i v1_i = \begin{bmatrix} v1_1 & v1_2 & \cdots & v1_{12} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{12} \end{bmatrix} = [\mathbf{V1}]\alpha. \tag{31}$$

Next, we impose a new constraint on solution space to reduce its dimensionality. In the previous section, we learnt that there exist infinite planes on which a coupler coordinate of a valid solution PR dyad can fall. However, for planar mechanisms, we are only interested in dyads which fall in the same plane as the poses. Thus, we consider solutions from only one plane constraint and constrain the solutions to $c_3 = 0$. Due to Eq. (12), this results in the following constraints on CSPs (P_i)

$$P_9 = P_{10} = P_{11} = P_{12} = 0 \tag{32}$$

To impose these constraints, we isolate coefficients from matrix $[\mathbf{V1}]$ in Eq. (31) as follows

$$\begin{bmatrix} P_9 \\ P_{10} \\ P_{11} \\ P_{12} \end{bmatrix} = \begin{bmatrix} v1_{1,9} & \cdots & v1_{12,9} \\ v1_{1,10} & \cdots & v1_{12,10} \\ v1_{1,11} & \cdots & v1_{12,11} \\ v1_{1,12} & \cdots & v1_{12,12} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{12} \end{bmatrix} = 0 \tag{33}$$

Applying these four constraints reduces the dimensionality of 12-D solution space by four. The system of equations in Eq. (33) is solved using SVD and the rightmost eight right singular vectors represent the solution α_i. These values are substituted back in Eq. (31) to find the new 8-D solution space given as

$$\mathbf{P} = \sum_{i=1}^{8} \beta_i v2_i = \begin{bmatrix} v2_1 & v2_2 & \cdots & v2_8 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_8 \end{bmatrix} = [\mathbf{V2}]\beta. \tag{34}$$

Finally, we need to apply the bi-linear constraints to the solution space. It should be noted there exist only six bi-linear constraints instead of nine since relationships between $P_9, P_{10}, P_{11}, P_{12}$ are automatically taken care of when constraining them to zero in Eq. (32). Thus, Eq (24) can be rewritten as

$$\begin{bmatrix} W_1 - \lambda_1 W_4 \\ W_2 - \lambda_2 W_4 \end{bmatrix}_{8\times 8} \beta_{8\times 1} = [\mathbf{W}]\beta = 0. \tag{35}$$

For this system of equations, we notice that the symbolic matrix $[\mathbf{W}]$ has a rank of 7. However, since five spatial pose problem has a 2-∞ solution space and we have enforced only one additional constraint of $c_3 = 0$, the rank of $[\mathbf{W}]$ should be six, giving us 1-∞ solutions. Thus, we set the determinant of two 7×7 sub-matrix to be zero and find the solution set for λ_1, λ_2. We can find upto 4 sets of λ_1, λ_2 which is in agreement common knowledge of planar five pose synthesis having up to four RR dyads.

These λ_i values are substituted back in Eq. (35) and their 2-D null space is calculated. The final solution 1-∞ CSPs can be calculated by substituting the null-space values back into Eq. (34) to get

$$\mathbf{P} = \begin{bmatrix} v3_1 & v3_2 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = [\mathbf{V3}]\gamma. \tag{36}$$

where γ_1, γ_2 are free homogeneous variables. Eq. (36) represents a set of spherical and planar constraints that can pass through the five input planar poses and whose coupler point lies on the same plane.

A circle constraint of a planar RR dyad can be represented as an intersection of Sphere and a Plane constraint. Thus, we need to find a plane and a circle in each solution space to find its RR dyad equivalent. To find equation of plane, the constraint $P_{16} = 0$ can be applied to find apt values of γ_1, γ_2. All other values represent a sphere and an arbitrary value can be selected.

Using the sphere and plane parameters, the nearest point on a plane from the center of sphere represents the Fixed Joint ($\mathbf{J}_F$) and is given by

$$\mathbf{J}_F = \mathbf{C} - k\mathbf{N}, \tag{37}$$

where $\mathbf{C}$ is the center of the circle, $\mathbf{N}$ is the normal vector of the plane and k is the minimum distance between Plane and center of sphere given as

$$k = \frac{Pl_1 Cr_1 + Pl_2 Cr_2 + Pl_3 Cr_3 + Pl_4}{Pl_1^2 + Pl_2^2 + Pl_3^2} \tag{38}$$

where a plane is given as $Pl_1 x + Pl_2 y + Pl_3 z + Pl_4 = 0$ and centre of circle is (Cr_1, Cr_2, Cr_3)

Thus, we can successfully find planar RR dyads using Spatial motion synthesis algorithm.

7. EXAMPLES

7.1 Example 1: Motion synthesis for seven general poses

To test our algorithm for accuracy we put it to test using the example from Innocenti's paper [19]. The pose quaternions are given in Table 1 and the twenty output dyads are given in Table 2 which match with the original results. The input poses are shown in Fig 3 and the spherical constraints in space are visualized in Fig 4. A 5-SS mechanism is created using smallest radius spheres and its geometric model is shown in Fig. 4. This 5-SS mechanism has been modelled in CAD software and shown in Fig. 1. The seven input poses can be seen on the trajectory the coupler point follows. As can be observed from Table 2, the output from proposed algorithm agrees exactly with results from Innocenti's paper.

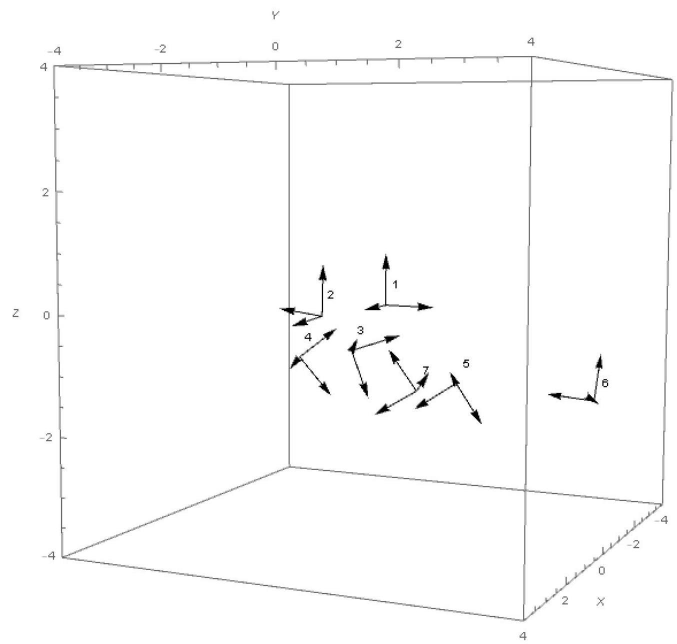


FIGURE 3: Input spatial poses for example 1

7.2 Example 2: Motion synthesis for seven planar poses

To validate the multiplicity of solutions in case where spatial poses lie on a plane, we choose seven poses as shown in Fig. 5 and given in Table 3. As discussed in the paper, we can see

TABLE 1: Seven Input Poses in Dual Quaternion representation for Example 1

q_1	q_2	q_3	q_4	g_1	g_2	g_3	g_4
0	0	0	1	0	0	0	0
-0.0651026	0.00725456	-0.75133	0.656667	0.607675	0.136295	-0.0644337	-0.0149826
-0.298121	-0.59615	0.526405	0.527851	-0.285992	-0.0998713	-0.379727	0.104369
-0.376624	0.859028	0.339364	0.0711109	0.138837	0.163292	-0.449864	0.90963
-0.582787	-0.346858	0.734635	0.018995	0.0351985	0.906884	0.447598	0.329131
0.00431816	-0.0844728	-0.561614	0.823064	-2.33102	0.1092	-0.690907	-0.448
0.821544	0.147805	0.372509	0.405532	0.205892	-0.485491	-0.539083	0.255028

TABLE 2: Synthesized Dyads for example 1

Dyad	A_1	A_2	A_3	A_4	C_1	C_2	C_3	C_4	r
1	-1.4532	-0.4131	-1.1781	1	1.3606	0.0849	-1.0281	1	2.8615
2	-0.3764	-0.2694	-2.255	1	2.1434	-0.9266	-0.1025	1	3.3786
3	-0.4049	-0.8841	-1.2398	1	1.6293	1.8376	-1.7463	1	3.4354
4	0.8106	-0.9742	-2.7163	1	2.3575	0.664	0.2455	1	3.7214
5	0.2611	2.4586	-3.4243	1	-1.5559	1.1521	0.2328	1	4.2874
6	0.9736	2.9071	-3.0424	1	-0.9779	1.0619	0.4361	1	4.3947
7	-0.1484	2.679	-0.4008	1	0.161	-0.6354	2.4776	1	4.4007
8	1.2796	0.716	-1.2142	1	-0.6459	4.1423	0.9059	1	4.4657
9	-3.4211	-0.2941	1.4625	1	0.7026	-0.3223	-0.6563	1	4.6363
10	-3.8199	-3.7259	4.3852	1	0.8758	2.4775	2.7771	1	7.9447
11	-2.5613	-4.1576	-8.7597	1	-0.3029	0.2047	-0.9218	1	9.25
12	-4.0709	-2.56	-3.6963	1	-1.3247	-7.2043	5.0687	1	10.2926
13	0.8993	-0.907	0.1315	1	3.4589	3.3523	-9.6642	1	10.984
14	-7.8388	-0.0889	9.6483	1	0.3671	-0.5878	-0.9344	1	13.4007
15	-7.7369	-9.6348	-10.4396	1	-0.4714	1.5845	-1.9814	1	15.8177
16	0.073	-0.5606	0.2413	1	-5.3932	3.2032	25.0842	1	25.7141
17	-3.2431	-34.9651	-7.2184	1	5.4831	-5.0918	14.7174	1	38.0754
18	-48.9413	-37.5477	-43.9745	1	-0.0682	5.1441	-4.516	1	75.9482
19	-7.9651	2.5179	-4.8166	1	51.361	26.9419	-62.1902	1	86.0688
20	75.4808	37.5827	-87.3605	1	-43.3276	-113.601	-109.994	1	193.608

multiple parallel planes as solution constraints in Fig. 6. The four dimensional output space is given in Table 4

7.3 Example 3: Motion synthesis for five planar poses which fall on a circle constraint

For this example, we use planar poses from an example used in Ge et al. [11]. The poses are given in Table 5 and have been plotted in Fig 7. As we have discussed before, we obtain four sets of solution spaces representing each planar RR dyad as shown in Table 6. Each of the four solution dyad is shown in Fig 8 and they match with the planar synthesis solutions. The final dyads have been visualized in Fig. 2a. As can be observed from Table 6, the output from proposed algorithm agrees exactly with results from the paper by Ge et al.

8. CONCLUSION

Thus, we have proposed a methodology to unify the spatial and planar motion synthesis problem. We use the algebraic fitting algorithm and intelligently constrain the solution spaces to find all possible RR dyads for planar four-bar mechanisms. In the future, we aim to combine the spherical synthesis framework with the existing framework to make a truly unified theory.

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TABLE 3: Input Planar Poses in Dual Quaternion representation for example 2

q_1	q_2	q_3	q_4	g_1	g_2	g_3	g_4
0.339875	0.170458	0.320228	0.867688	0.941997	0.465082	-0.430818	-0.30135
0.289649	0.246322	0.104923	0.918924	-2.05708	1.26046	0.67629	0.23331
0.35765	0.129061	0.42061	0.823721	0.128246	1.45804	0.106495	-0.338508
0.369115	0.091241	0.503837	0.775614	0.566952	1.34614	-0.0411549	-0.401435
0.369712	0.0887888	0.508975	0.772253	-0.666564	-1.92782	-0.589628	0.929375
0.344994	0.159842	0.346966	0.857347	-0.637635	2.33142	0.570922	-0.409134
0.371241	0.0821652	0.522699	0.76303	-0.522412	2.96078	0.743448	-0.573938

TABLE 4: Synthesized CSP Solution space for example 2

P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
0.547858	-0.212638	0.199037	-0.294373	0	-0.230202	0.215478	-0.318688	0
-0.694203	-0.276618	0.258925	-0.382946	0	-0.171187	0.160238	-0.236989	0
-0.104699	0.32232	-0.301704	0.446216	0	-0.395045	0.369777	-0.546894	0
0.214242	-0.195044	0.182569	-0.270017	0	-0.159081	0.148906	-0.220229	0
P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	
-0.236481	0.221356	-0.327381	0	-0.174342	0.163191	-0.241357	0	
0.0984027	-0.0921086	0.136227	0	-0.14524	0.13595	-0.201068	0	
0.0285454	-0.0267196	0.0395178	0	0.00302892	-0.00283519	0.0041932	0	
0.165428	-0.154847	0.229016	0	0.401058	-0.375406	0.555219	0	

TABLE 5: Input Planar Poses in Dual Quaternion representation for example 3

q_1	q_2	q_3	q_4	g_1	g_2	g_3	g_4
0.18182	0.0208603	0.779558	0.598996	-0.694984	-0.283932	0.177855	-0.010623
0.173037	0.0595978	0.63196	0.753083	-0.811229	0.0912475	0.256593	-0.0361472
0.162258	0.084653	0.513016	0.838642	-1.18682	0.385687	0.358205	-0.0284309
0.157636	0.0929757	0.468714	0.864184	-1.61222	0.444565	0.435748	0.00991535
0.158956	0.0907013	0.481082	0.85736	-1.93825	0.297065	0.472565	0.0627621

TABLE 6: Synthesized four planar RR dyadic parameters for example 3

A_1	A_2	A_3	A_4	C_1	C_2	C_3	C_4	r
2.35097	0.0675858	-0.438144	1.0000	6.45577	6.50408	0	1.0000	11.199
1.20255	0.348633	-0.0026419	1.0000	4.58995	1.34007	0	1.0000	4.6712
2.73061	2.27873	-0.124526	1.0000	5.03163	-7.40889	0	1.0000	4.06469
-1.87116	-0.754965	0.794402	1.0000	1.24	0.1	0	1.0000	1.23773

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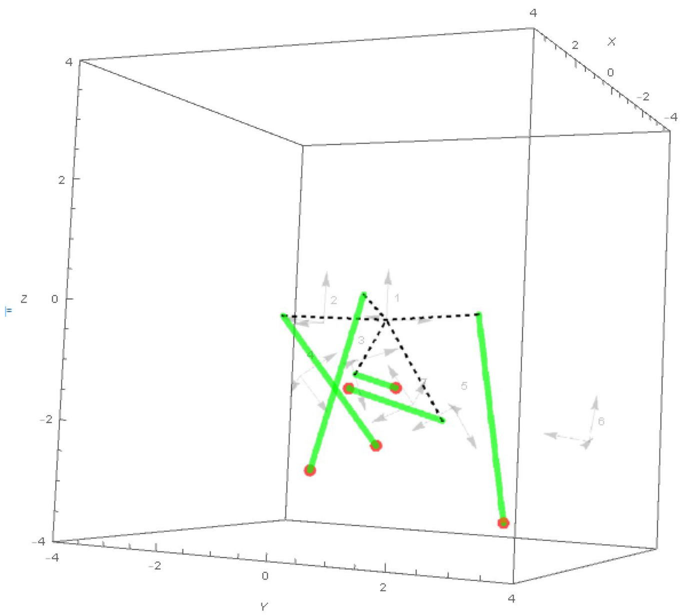

FIGURE 4: Five of the total twenty output spherical constraints for example 1

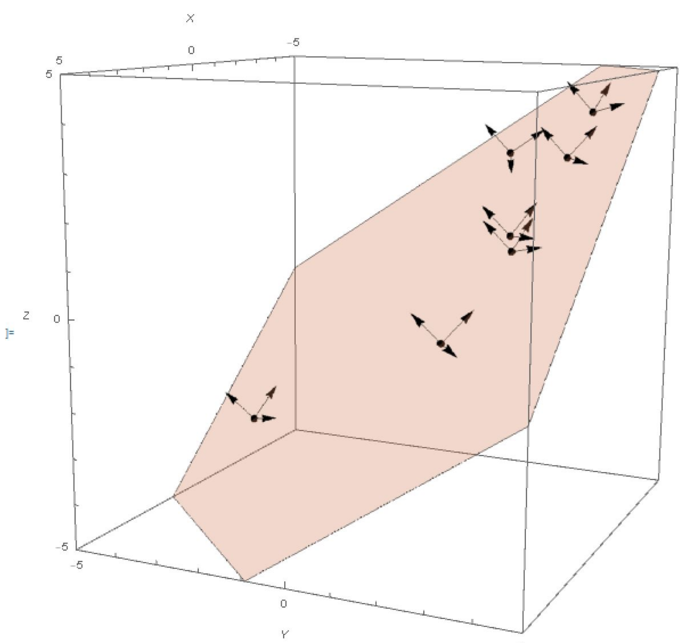

FIGURE 5: Input spatial poses for example 2

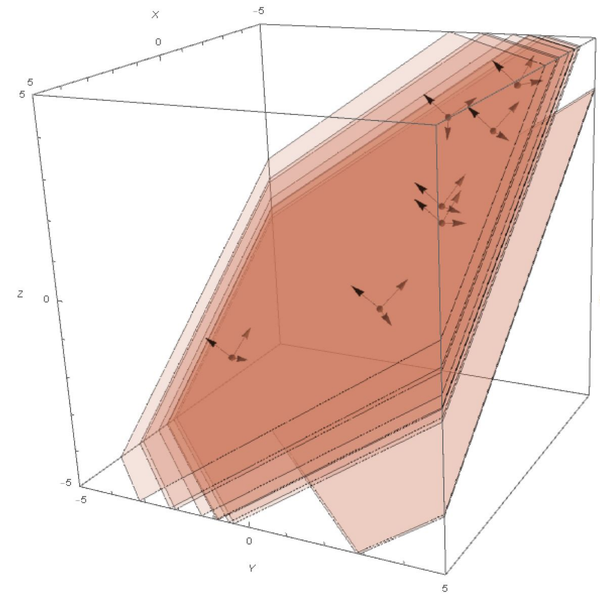

FIGURE 6: Output planar constraints for example 2

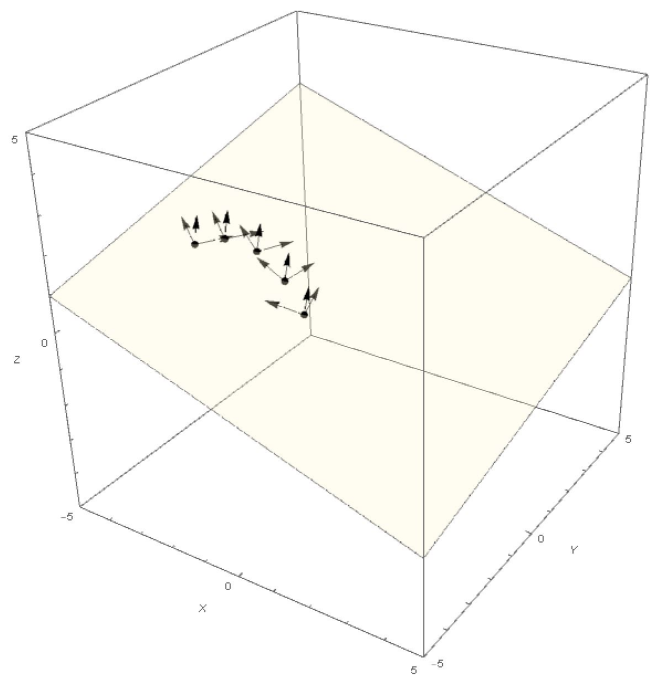

FIGURE 7: Input spatial poses for example 3

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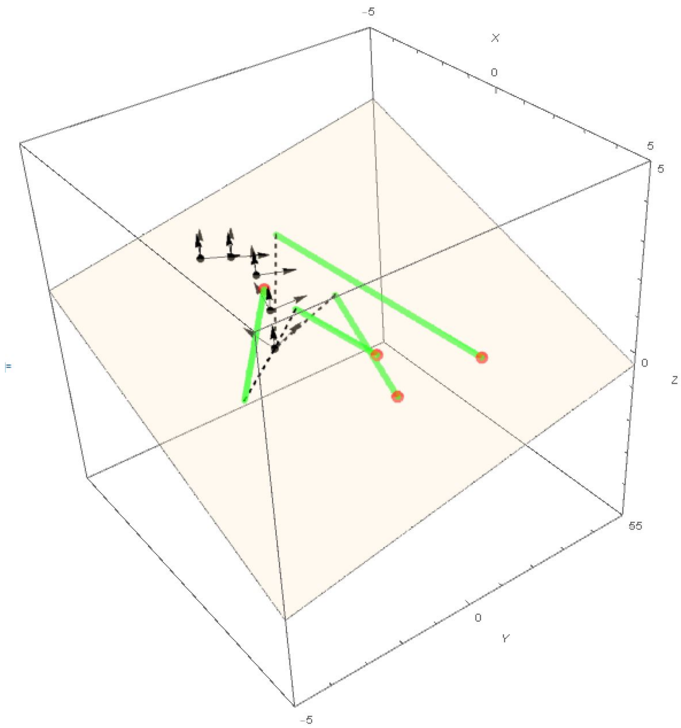

FIGURE 8: Set of four RR dyad solutions for example 3

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