An Optimal Parametrization Scheme for Path Generation Using Fourier Descriptors for Four-Bar Mechanism Synthesis

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Fourier descriptor (FD)-based path synthesis algorithms for generation of planar four-bar mechanisms require assigning time parameter values to the given points along the path. An improper selection of time parameters leads to poor fitting of the given path and suboptimal four-bar mechanisms while also ignoring a host of mechanisms that could be potentially generated otherwise. A common approach taken is to use uniform time parameter values, which does not take into account the unique harmonic properties of the coupler path. In this paper, we are presenting a nonuniform parametrization scheme in conjunction with an objective function that provides a better fit, leverages the harmonics of the four-bar coupler, and allows imposing additional user-specified constraints. [DOI: 10.1115/1.4041566]

Keywords: Planar four-bar linkage, open and closed path generation, nonuniform Fourier descriptor, optimal parametrization, harmonic analysis

1 Introduction

This paper is concerned with the path generation problem, wherein a path is usually given as a sequence of discrete points in R^2 and the goal is to find dimensions of a planar four-bar mechanism such that a point on the coupler of the mechanism traces the given points as close as possible [1]. Optimization-based techniques attempt to minimize an objective function and find mechanisms, which best approximate a curve generated by using Fourier series [2]. This curve, called the task curve, requires point data on the prescribed path and associated time parameter values. Discrete Fourier transform has been used to calculate the Fourier coefficients or Fourier descriptors (FDs) of the task curve from the prescribed path. FDs have been frequently used in computational shape analysis to create the objective function [3–9]. This paper focuses on solving the path generation problem using an FD-based technique.

McGarva and Mullineux [5] studied the inherent dependency of FDs of a task curve on time parametrization and concluded that different parametrization leads to different FDs. In previous studies, this limitation has been ignored and the parametrization has been assumed to be uniform. This results in a task curve, which might be suboptimal for use in mechanism synthesis.

Figure 1 demonstrates an instance, where using a nonuniform parametrization yields a better mechanism than a uniform parametrization. A test mechanism is taken and ten arbitrary points are sampled on its coupler curve to generate the input data. Once the task curve is calculated, a four-bar coupler curve is fitted to synthesize a mechanism for each case. Comparing the resulting mechanisms with the input shows that nonuniform parametrization fits the points more accurately and better matches the first mechanism. Although, in computer aided design, finding parametrization for a given sequence of points in the context of curve interpolation is a common problem, here we have the additional burden of ensuring that the parametrization is compatible with the properties of the coupler curves of planar four-bar mechanisms. In the example presented later on, we will show that the optimized task curve matches the harmonic contents of the coupler curve better than an arbitrary choice of parameters. Finding this optimum nonuniform parametrization serves as the motivation of this paper.

Recently, Li and Chen [10] proposed an approach to eliminate time dependency using arc length parametrization. It is well known in computer aided design community that finding an explicit closed-form expression for arc-length parametrization is impossible. As a result, they numerically *guess* the nonuniform parametrization using the distance between input points. Geometrically, the formulated parametrization is close to being time-independent. But this approach forces the coupler point to move at a constant speed along the task curves and eliminates a host of other possible four-bar mechanisms.

In this paper, a novel methodology, which calculates the optimal parametrization for a sequence of points, has been proposed. For a four-bar coupler curve, the magnitude of its higher order harmonics has been observed to be negligible by Freudenstein [3] and Li et al. [11]. This property is used to search for an optimal parametrization to generate a task curve with low magnitude higher order harmonics. Coupler speed criteria for enhanced control over the task curve have also been incorporated. A new cost function is proposed that combines the cost of fitting to the given data points, the cost of low magnitude higher order harmonic, and the penalty for larger speed ratios. Nelder–Mead optimization is used to compute two critical state space parameters that minimize the cost function and provide optimized time parameters. Thereafter, we use the algorithm presented by Wu et al. [7] to synthesize a four-bar mechanism from the optimal task path. This algorithm fits task curve FDs to coupler curve FDs using a four-dimensional search space instead of the conventional ten-dimensional search space.

The rest of this paper is organized as follows: Section 2 reviews an existing FD-based path generation approach, which supplements the proposed algorithm for four-bar mechanism synthesis. Section 3 introduces a family of nonuniform parametrization and a methodology for finding the optimal one among them. Section 4 presents an example, which illustrates the effectiveness of the proposed approach.

2 Review of Fourier Descriptor Based Path Generation

This section provides a brief overview of an existing FD-based path generation algorithm proposed by Wu et al. [7], which is used in conjunction with the improved task curve generation method as discussed in Sec. 3 for four-bar synthesis.

In this method, input path points are used to calculate a task curve described by a trigonometric polynomial curve with an open interval and is represented as

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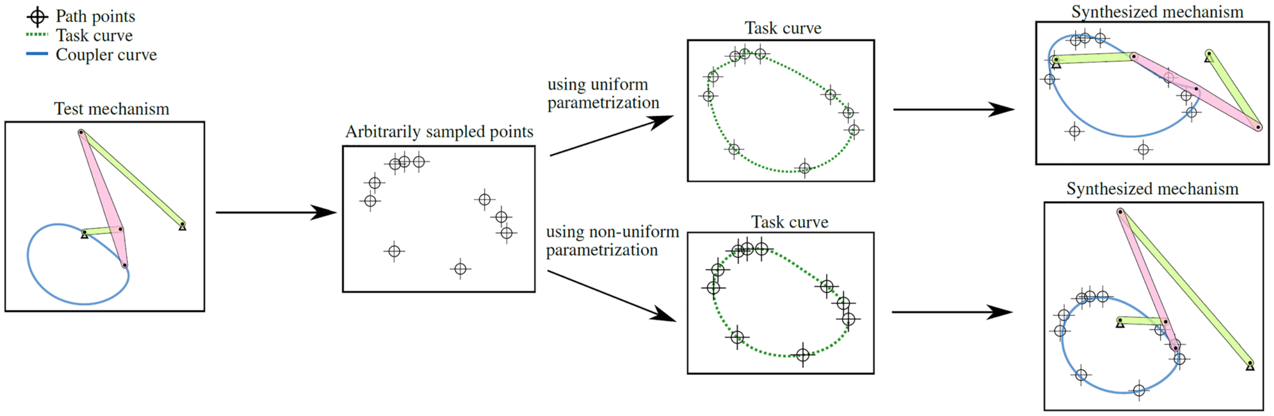


Fig. 1 Path generation of two four-bar mechanisms; one using uniform parametrization while the other using optimal nonuniform parametrization

$$z(t) = \sum_{k=-p}^{p} T_k e^{ik\omega_o t} \qquad \forall \quad t \in [0, t_{\max}] \tag{1}$$

where $z(t) = x(t) + iy(t)$ denotes the point coordinates in complex form at time t, k are the frequency indices, T_k are the FDs, ω_o is the angular velocity of crank, and [0, $t_{\max}$] is the time interval over which the curve is defined. It has been shown that a task curve with $p = 5$ captures the four-bar coupler path very accurately and is deemed sufficient for practical implementation [11]. Taking $\omega_0 = 2\pi$ and $t_{\max} \in (0, 1]$ represents the possible closed and open task paths. The FDs are calculated by solving the least square fitting problem with the objective function as

$$\Delta = \sum_{i=1}^{n} \left\| z(t_i) - \sum_{k=-p}^{p} T_k e^{ik\omega_o t_i} \right\|^2 \tag{2}$$

where n is the number of input points. If $n < (2p+1)$, then the highest $2\lceil (2p+1-n)/2 \rceil$ harmonics are specified as zero to find a unique solution. Here, $\lceil \ldots \rceil$ represents the ceiling function. Finding the domain [0, $t_{\max}$] for the open task path is a one-dimensional optimization problem for minimum error measure defined in Eq. (2). Once a time parametrization is chosen or assumed, the task curve can be calculated.

A four-bar mechanism can be represented by the design parameters x_0, y_0, l_1, l_2, l_3, l_4, r, α, θ_1, and initial crank angle ϕ_0 as displayed in Fig. 2. The analytic equation of coupler point P can be given as

$$P = A_0 + l_2 e^{i\theta_1} e^{i\phi} + r e^{i\alpha} e^{i\theta_1} e^{i\lambda} \tag{3}$$

where λ is the coupler angle as shown in Fig. 2 and A_0 is the complex form of the fixed pivot given by (x_0, y_0). A closed-form expression for λ using loop-closure criteria is available in literature [7], and can be approximated using FDs as

$$e^{i\lambda} = \sum_{k=-p}^{p} C_k e^{ik\phi} \tag{4}$$

where C_k are coupler angle FDs whose value depends upon the link lengths l_1, l_2, l_3 and l_4, and $\phi = \phi_0 + \omega_0 t$.

The coupler path can also be expressed as a Fourier series given in Eq. (5) where P_k are coupler path FDs

$$P = \sum_{k=-p}^{p} P_k e^{ik\omega t} \tag{5}$$

By substituting Eq. (4) into Eq. (3) and collecting coefficients of $e^{ik\omega t}$ to form Eq. (5), we can express the P_k as

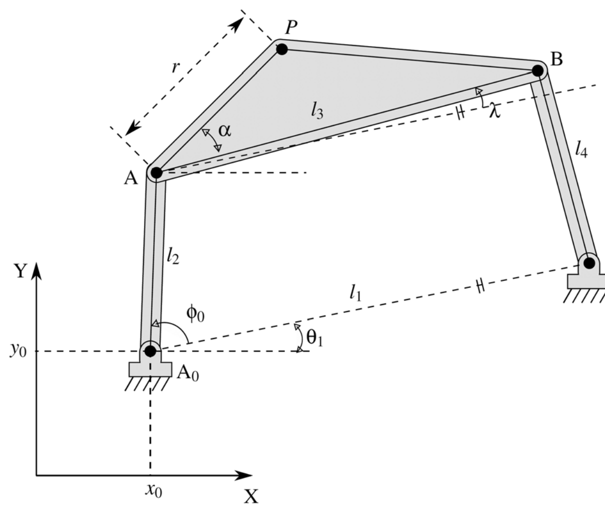


Fig. 2 A planar four-bar mechanism showing dimensional parameters

$$P_0 = r e^{i\alpha} e^{i\theta_1} C_0 + A_0 \tag{6}$$

$$P_1 = r e^{i\alpha} e^{i\theta_1} C_1 e^{i\phi_0} + l_2 e^{i(\theta_1 + \phi_0)}, \quad \text{and} \tag{7}$$

$$P_k = r e^{i\alpha} e^{i\theta_1} C_k e^{ik\phi_0}\big|_{k \neq 0,1} \tag{8}$$

The coupler path can now be fitted to the task curve to calculate the four-bar design parameters using Eqs. (1) and (5). Equating T_k to P_k leads to a system of equations with ten unknowns given as following:

$$S = \left\{ l_2, \frac{l_2}{l_1}, \frac{l_3}{l_1}, \frac{l_4}{l_1}, x_0, y_0, \theta_1, \phi_0, \mathbb{C}, \mathbb{S} \right\} \tag{9}$$

where $\mathbb{C} = r\cos(\alpha + \theta_1)$ and $\mathbb{S} = r\sin(\alpha + \theta_1)$. Equation (8) depends on six design variables $\{l_2/l_1, l_3/l_1, l_4/l_1, \phi_0, \mathbb{C}, \mathbb{S}\}$, while the remaining four variables $\{l_2, x_0, y_0, \theta_1\}$ exist independently in Eqs. (6) and (7). Wu et al. [7] show that the six-dimensional design space in Eq. (8) can be further reduced to a four-dimensional space of $\{l_2/l_1, l_3/l_1, l_4/l_1, \phi_0\}$ by analytically minimizing the following objective function:

$$I = \sum_{k \neq 0,1} |C_k r e^{i(\alpha + \theta_1 + k\phi_0)} - T_k|^2 \tag{10}$$

This objective function is obtained by summing the squared difference of P_k given in Eq. (8) and T_k.

The direct search method has been used by Wu et al. [7] to solve this optimization problem. In summary, matching task path FDs to coupler path FDs using a four-dimensional search space forms the core of this approach.

3 Optimum Parametrization

The task curve calculation is inherently associated with the time parametrization used. However, there are infinite ways to select a nonuniform parametrization. To make the problem tractable and facilitate the selection of a single nonuniform parametrization, a chord length-based parametrization scheme for n-points sequence is defined as

$$t_1 = 0 \tag{11}$$

$$t_k = t_{\max}\left(\frac{\sum_{i=2}^{k}|z_i - z_{i-1}|^{\alpha}}{\sum_{i=2}^{n}|z_i - z_{i-1}|^{\alpha}}\right) \tag{12}$$

Here, t_k represents the time parameter associated with the kth path point, $t_{\max}$ is the interval domain as defined in Eq. (1), z_i represents the coordinates of ith point, and $\alpha \in \mathbb{R}$ is the parametrization control variable. Varying the control variable α generates multiple parametrizations. In this scheme, when $\alpha = 0$ and $\alpha = 1$, we obtain uniform and arc length parametrizations, respectively. Physically, uniform parametrization results in a task curve where the coupler takes equal amount of time to pass through each target point. Similarly, the arc length parametrization approximates constant speed motion of the coupler. By varying α, one can generate different parametrizations for the calculation of the task curves, which, in turn, could provide a range of mechanism design solutions and also facilitates selection of an optimal parametrization, leading to mechanisms that provide better fit with the input data.

To measure the quality of the task curve, we define a cost function as following:

$$C_t = C_f + C_h + C_s \tag{13}$$

where C_f is the cost attributed to FD fitting error, C_h is the cost due to higher order harmonic content, and C_s is the cost due to enforced speed criteria on the task curve.

If the path (z_i), parametrization control variable (α), and time domain ($t_{\max}$) are known, the FDs of an open task curve can be mean square fitted as shown in Eq. (2). The square-root of the residual in the fitting process is normalized to define C_f in the cost function and is given as

$$C_f = \frac{1}{n}\sqrt{\sum_{i=1}^{n}\left\| z_i - \sum_{k=-p}^{p} T_k e^{ik\omega_o t_i}\right\|^2} \tag{14}$$

Here, n represents the total path points, z_i are the path coordinates, T_k are the task curve FDs, and t_i is the time parameter attached to ith point. This term ensures that the selected parametrization and domain accurately represent the original point data.

The second term in the cost function, namely harmonic cost, is motivated by the observation from Li et al. [11] that the magnitude of higher order harmonics is negligible for four bars. As a result, task curves whose higher order harmonics have minimal magnitude would be better prospective curves for the path synthesis process. To enforce this harmonic criterion, a weighted harmonic magnitude-based metric is defined as

$$C_h = \frac{1}{2p+1}\sum_{k=-p}^{p}(|k|+1)^{\beta}\|T_k\| \tag{15}$$

Here, p is the maximum number of harmonics being considered, $\beta \geq 1$ is a constant, and T_k are the task curve FDs. The term $(|k|+1)^{\beta}$ adds larger weight to the higher order harmonics.

Table 1 Input point data

No.	Coordinate (x, y)	No.	Coordinate (x, y)	No.	Coordinate (x, y)
1	0.000, −1.000	5	−2.866, −2.118	9	0.246, −2.135
2	−0.550, −0.942	6	−2.608, −2.488	10	0.876, −1.615
3	−1.696, −1.018	7	−2.098, −2.720	11	0.986, −1.329
4	−2.821, −1.715	8	−0.546, −2.551	12	0.593, −1.123

In practical scenarios, large coupler speed changes can lead to large induced forces on the links, which could compromise their rigidity and render the kinematic analysis useless. In contrast, a uniform speed motion of the coupler can also be undesirable in some instances, such as when designing a quick-return mechanism. Thus, enforcing a speed-based criterion over the task curve gives users more control in design. To select a task curve with desirable speed properties, speed cost (C_s) is defined using quadratic penalty function as follows:

$$C_s = w[\max(0, S_{r,\min} - S_r)^2 + \max(0, S_r - S_{r,\max})^2] \tag{16}$$

where S_r represents ratio of task curve's maximum to minimum speed. $S_{r,\min}$ and $S_{r,\max}$ are the minimum and maximum speed ratio bounds enforced by the designer and w is the penalty weight imposed. In case there are no speed restrictions, $S_{r,\min}$ and $S_{r,\max}$ can be set to 1 and infinity, respectively. $w = 10^3$ has been taken in the implementation. The expression for task curve speed can be calculated by differentiating Eq. (1), which gives

$$s(t) = \left\|\sum_{k=-p}^{p} ik\omega_o T_k e^{ik\omega_o t}\right\| \tag{17}$$

Maximum and minimum speeds can be calculated numerically by sampling a large number of points over the task curve.

Thus, an optimal task curve can be calculated by minimizing the total cost (C_t) as given in Eq. (13). The search space is two-dimensional with α and $t_{\max}$ as the state variables. Thereafter, the optimal chord length-based time parametrization can be calculated using Eqs. (11) and (12). Nelder–Mead optimization has been used for searching the state space. With the task curve known, a four-bar mechanism can be generated as discussed in Sec. 2. The complete Fourier-based path synthesis algorithm using optimum parametrization has been summarized in the Algorithm 1.

Algorithm 1: Path generation using Fourier descriptor based approach with optimal parametrization

Input: Set of path points
1 Search for optimum α and $t_{\max}$ by minimizing C_t given in Eq. (13).
2 Calculate $\left\{\frac{l_2}{l_1}, \frac{l_3}{l_1}, \frac{l_4}{l_1}, \phi_0, \mathbb{C}, \mathbb{S}\right\}$ by minimizing I given in Eq. (10)
3 Calculate $\{l_2, x_0, y_0, \theta_1\}$ using Eqs. (6) and (7) to synthesize a four-bar mechanism.
Output: Four-bar design parameters

4 Example

This example demonstrates the improvement made using the proposed methodology. A 12-point trajectory is taken as the input and is given in Table 1. In this example, we use $\omega_o = 2\pi$ rad/s (Eq. (2)) and $\beta = 2$ (Eq. (15)).

First, we use uniform parametrization to create a task curve. The task curve is calculated to have $t_{\max} = 0.8770$ and FDs as given in Table 2. The task curve fitting error, as defined in Eq. (2), is found to be $\Delta = 0.3509$. From this task curve, a mechanism is synthesized to find the four-bar design parameters. Coupler curve FDs and computed mechanism parameters are given in Tables 3 and 4, respectively. Coupler curve fitting error, as defined in

Table 2 Task curve FDs

Descriptor	Parametrization: Uniform	Optimal	Optimal with speed criteria
−5	$0.035-0.051i$	$-0.001+0.013i$	$-0.003+0.017i$
−4	$-0.021+0.033i$	$-0.034-0.013i$	$-0.011-0.007i$
−3	$0.008+0.037i$	$-0.022-0.014i$	$-0.021-0.016i$
−2	$-0.015+0.086i$	$-0.009-0.018i$	$-0.003-0.036i$
−1	$0.420-0.455i$	$0.412-0.290i$	$0.378-0.261i$
0	$-0.822-1.696i$	$-0.960-1.763i$	$-0.970-1.764i$
1	$0.352+1.334i$	$0.715+1.167i$	$0.719+1.148i$
2	$0.074-0.204i$	$-0.005-0.072i$	$-0.015-0.065i$
3	$0.054-0.051i$	$-0.053-0.008i$	$-0.072+0.013i$
4	$-0.051-0.077i$	$-0.045+0.002i$	$-0.021-0.002i$
5	$-0.019+0.042i$	$-0.002-0.010i$	$0.003-0.032i$

Table 3 Coupler curve FDs

Descriptor	Parametrization: Uniform	Optimal	Optimal with speed criteria
−5	$0.011+0.006i$	$-0.001+0.000i$	$-0.001+0.000i$
−4	$0.008+0.019i$	$0.000+0.001i$	$-0.001+0.000i$
−3	$0.049-0.019i$	$-0.006-0.011i$	$-0.007-0.014i$
−2	$0.005+0.104i$	$-0.005-0.020i$	$-0.001-0.031i$
−1	$0.430-0.453i$	$0.412-0.291i$	$0.376-0.262i$
0	$-0.822-1.696i$	$-0.960-1.763i$	$-0.970-1.764i$
1	$0.352+1.334i$	$0.715+1.167i$	$0.719+1.148i$
2	$0.055-0.170i$	$0.004-0.074i$	$-0.006-0.077i$
3	$0.000-0.034i$	$-0.006+0.006i$	$-0.004+0.007i$
4	$-0.006-0.025i$	$0.003-0.000i$	$0.003-0.001i$
5	$-0.005-0.012i$	$-0.001-0.000i$	$-0.001-0.000i$

Eq. (10), is calculated to be $I=0.0228$. The synthesized mechanism can be viewed in Fig. 3(*a*).

Next, mechanism synthesis is accomplished using optimal parametrization. The task curve is calculated to have $t_{\max}=0.9336$, $\alpha=0.5823$ and FDs as given in Table 2. The task curve fitting error is observed to be $\Delta=0.0591$, which is better than the previous case. Subsequently, a mechanism is synthesized and the coupler curve FDs and the solution mechanism parameters are given in Tables 3 and 4, respectively. The coupler curve fitting error is

Table 4 Synthesized mechanism parameters

Variable	Parametrization: Uniform	Optimal	Optimal with speed criteria
l_1	10.020	9.271	10.140
l_2	1.921	2.085	1.464
l_3	6.719	6.520	3.994
l_4	5.423	5.160	8.139
x_0	−6.464	−5.560	11.951
y_0	1.659	0.931	−2.386
θ_1	−0.346	−0.191	0.859
r	12.191	11.519	10.194
α	0.119	0.001	0.661
ϕ_0	0.774	0.764	3.218

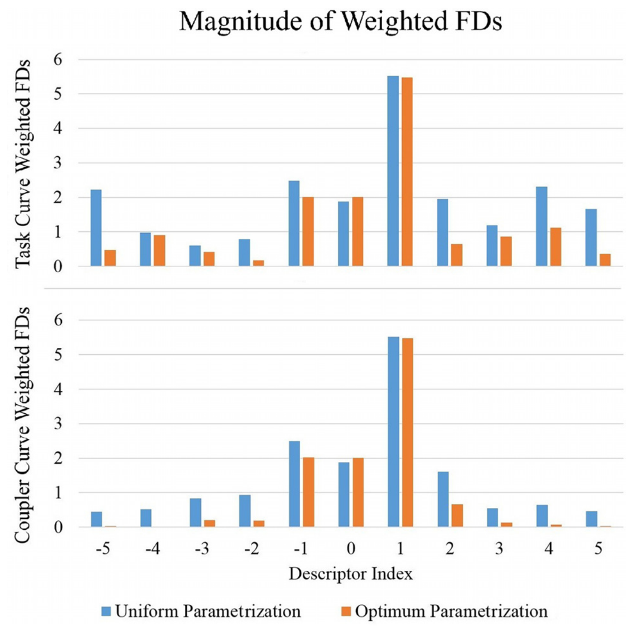
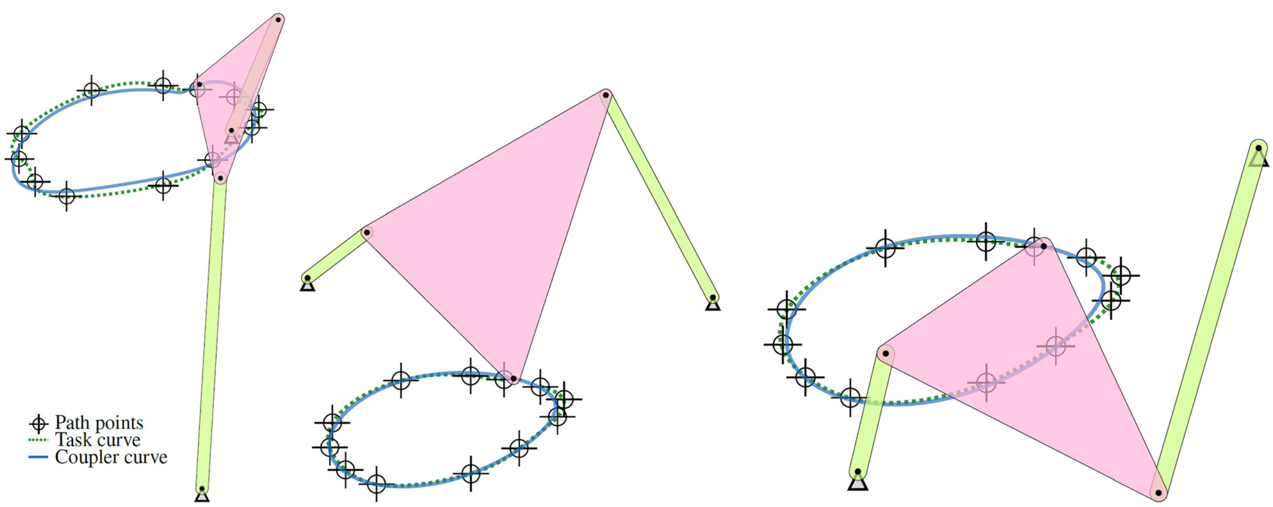


Fig. 4 Comparison of task curve and coupler curve weighted FDs for uniform and optimal parametrization



(*a*) Uniform parametrization (*b*) Optimal parametrization (*c*) Optimal parametrization with speed criteria

Fig. 3 Synthesized solutions using different parametrizations: (*a*) uniform parametrization, (*b*) optimal parametrization, and (*c*) optimal parametrization with speed criteria

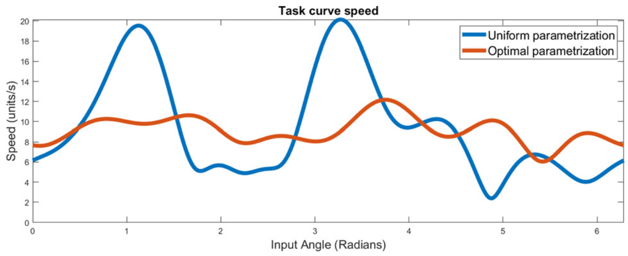


Fig. 5 Comparison of task curve speeds obtained with and without speed criteria

found to be $I = 0.0067$, which is also less than the uniform parametrization case. The synthesized mechanism can be viewed in Fig. 3(*b*). While using nonuniform parametrization enables the reduction of fitting error Δ, this example demonstrates that a task curve with low magnitude higher order harmonics decreases I and leads to a better task-coupler curve matching. Figure 4 displays the comparison of weighted FDs, given as $T_{w,k} = (|k|+1)^2||T_k||$, for uniform and optimal parametrization. These figures show that for optimal parametrization case, the magnitude of the higher order harmonics is less compared to the uniform case for both the task and the coupler curves.

Finally, mechanism synthesis involving speed criteria is carried out. Speed ratio for the task curve calculated using uniform parametrization is observed to be 8.22. Let us assume that the user desires to constrain S_r such that $1 \leq S_r \leq 2$. After applying the speed criteria, the task curve is calculated to have $t_{\max} = 0.9234$, $\alpha = 0.7885$ and FDs as given in Table 2. The task curve fitting error is observed to be $\Delta = 0.1813$. The S_r of the generated task curve is 2. Comparison of task curve speeds has been done in Fig. 5. It can be observed that the new task curve has reduced the speed ratio. Synthesized mechanism design parameters are given in Table 4. The coupler curve fitting error is observed to be $I = 0.0444$ and the solution is displayed in Fig. 3(*c*).

5 Conclusion

In this paper, a nonuniform parametrization scheme has been proposed for the task curve calculation from a given sequence of path points. A novel methodology to find the optimal parametrization based on fitting accuracy, the harmonic properties of four-bar coupler path, and user imposed speed criteria have been demonstrated. Synthesis of a more accurate four-bar mechanism for path generation has been shown using an example. The proposed approach improves upon the existing FD based path generation algorithm for mechanism synthesis.

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