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A Motion Synthesis Approach to Solving Alt-Burmester Problem by Exploiting Fourier Descriptor Relationship Between Path and Orientation Data

This paper presents a generalized framework to solve m*-pose,* n*-path-points mixed synthesis problems, known as the Alt-Burmester problems, using a task-driven motion synthesis approach. We aim to unify the path and motion synthesis problems into an approximate mixed synthesis framework. Fourier descriptors are used to establish a closed-form relationship between the path and orientation data. This relationship is then exploited to formulate mixed synthesis problems into pure motion synthesis ones. We use an efficient algebraic fitting based motion synthesis algorithm to enable simultaneous type and dimensional synthesis of planar four-bar linkages.* [DOI: 10.1115/1.4042054]

Keywords: mechanism synthesis, theoretical kinematics

1 Introduction

Conventionally, mechanism synthesis problems have been categorized and studied independently as path, motion, and function synthesis problems [1]. Path synthesis problems specify only path-point coordinates (x_i, y_i), while motion synthesis problems specify pose constraints (x_i, y_i, ζ_i), where (x_i, y_i) are the coordinates of the path points or the origin of a moving frame attached to a given pose, while ζ_i is the orientation of the moving frame. In function synthesis, only input–output angle pairs (θ_i, ψ_i) are specified. Unfortunately, most of the real world problems do not conform to such a rigid categorization—many practical problems provide a mixture of path, motion, and function synthesis requirements. However, a synthesis approach which seamlessly incorporates all the three conventional synthesis problems has been elusive. As a result, machine designers often have to compromise on design specifications. In this paper, the focus is on the synthesis of planar four-bar mechanisms for a hybrid of path and motion synthesis problems. This problem formulation, which consolidates both path and orientation data, has been termed as mixed synthesis in this paper.

Murray's group termed the combined path and motion problems as the Alt-Burmester problems [2] named after Alt's [3] and Burmester's [4] work on path and motion generation, respectively. Brake et al. [5] discuss the dimensionality of solution sets for a variety of path-point and pose combinations. However, a finite number of solutions exist only for a subset of possible m-pose, n-path-point synthesis problem. For example, there exist finite solutions for nine path points and for five poses independently. We define such problems to be fully constrained problems. For a lesser number of path points or poses, usually an infinite number of solutions are obtained. Subsequently, the authors explore only fully constrained or under-constrained problem sets where up to nine constraints can be used to find four-bar mechanism parameters. This ignores the vast majority of over-constrained problems in m-pose, n-path-point mixed synthesis family of problems, where exact solutions are not possible and only approximate, albeit useful solutions, can still be obtained. This is reflective of real-world design problems, which usually impose a large number of often challenging constraints.

A graphical approach has been presented by Zimmerman [6] to solve the mixed path, motion and function problem using sketching tools built in modern computer-aided design (CAD) softwares. The proposed methodology can conveniently solve under-constrained and fully constrained mixed synthesis problems and generate four-bar mechanisms. However, this methodology is unable to solve generalized m pose, n path-point synthesis problems, which may be over-constrained. This study does state the possibility of including prismatic joints in the synthesized mechanism. However, it is not the focus of study and details on synthesizing mechanical dyads with at least one prismatic joint are not included.

Motion synthesis turns out to be a mathematically less complex problem than path synthesis as each dyad can be generated independently effectively halving the number of unknowns. Typically, path synthesis problems involve solving a nonlinear system of equations. We have recently presented a generalized framework for solving motion synthesis problems using an efficient algorithm that involves solving a linear system of equations using singular value decomposition (SVD) [7–11]. The algorithm produces multiple solutions and can compute both the type and dimensions of the four-bar mechanisms. The algorithm produces results in real time and is thus amenable to its implementation in interactive computational design tools [7].

In this paper, we are presenting an approach to solve the Alt-Burmester problem by reducing it to a pure motion synthesis problem so that the aforementioned algorithm can be leveraged. In a planar four-bar linkage, the path of a coupler point is inextricably tied to the orientation of the coupler. This coupling can be revealed by analyzing and relating the harmonic content of the path and orientation data. First, an analytical relationship between the orientation- and path data is obtained using the harmonic breakdown of the loop closure equation. Then, this relation is used to reformulate the mixed synthesis problem into a motion synthesis problem by attaching compatible orientations to input path points and consequently turning them into poses. The Fourier approximation-based analytical approach proposed in this paper can handle almost all possible variations of path points or poses. Once, the problem has been converted into a pure motion

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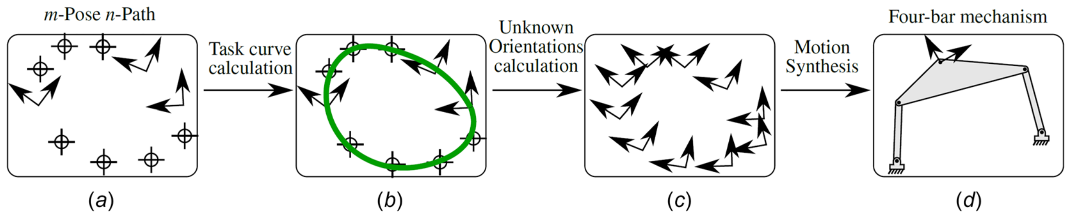

Fig. 1 An overview of our approach to the Alt-Burmester problems: (*a*) specify *m*-pose, *n*-path points, (*b*) a task curve is fit through the $m+n$ path points using Fourier series, (*c*) use the harmonic content of the path data to find the *missing* orientations at the *n*-path points, and (*d*) finally, compute both type and dimensions of planar four-bar linkages

synthesis problem, we re-purpose our algebraic fitting approach in Refs. [10] and [11] to solve for four-bar linkages. Figure 1 provides an overview of this approach.

We note that here mixed synthesis does *not* refer to the mixed exact-approximate path or motion synthesis, wherein we have a set of precision and approximate constraints. Our definition of *mixed* refers to a mixture of path-point and pose constraints.

This paper's original contributions are in (1) the formulation of a Fourier descriptor-based closed-form relationship between coupler orientations and path, (2) the novel use of this relationship to solve the generalized *m*-pose, *n*-path mixed synthesis problem, and (3) the incorporation of task-driven algebraic fitting-based motion synthesis within the mixed synthesis algorithm for synthesis.

Rest of the paper is organized as follows: Section 2 calculates a new path–orientation formulation from existing four-bar loop closure Fourier decomposition. Section 3 discusses the use of path–orientation relationship to reformulate mixed synthesis into motion synthesis problem. Section 4 reviews algebraic fitting-based motion synthesis algorithm. Section 5 proposes a new algorithm to solve mixed synthesis problem and finally in Sec. 6, we present a few examples to demonstrate the efficacy of the proposed approach.

2 Fourier Descriptor-Based Relations

Use of Fourier descriptors is abundant in the domain of mechanism synthesis. It has been used for planar four-bar mechanism synthesis using optimization routines [12–16], atlas-based search algorithms [17,18], and machine learning approach [19]. Fourier descriptors have also been used to synthesize spherical [20] and spatial mechanism [21]. A class of single degree-of-freedom open-loop mechanisms termed as planar coupled serial chain mechanisms [22,23] have also been generated with the help of Fourier descriptors.

In this section, we are interested in exploring the relationship between the coupler path and coupler orientation to establish a closed-form relationship between them. This would give us a framework for dealing with both pose and path constraints simultaneously. Path and motion synthesis formulations, which use Fourier decomposition of four-bar closure equation [14–16] are used as a starting point here. In Ref. [16], Li et al. presented a decomposition of the design space of four-bar mechanisms by using Fourier descriptors in the context of planar motion approximation. Harmonic decomposition of four-bar loop closure equation has been analyzed to independently fit rotational and translational Fourier descriptors and synthesize motion.

A four-bar mechanism is represented by its design parameters x_0, y_0, l_1, l_2, l_3, l_4, r, θ_1, and α as displayed in Fig. 2. These parameters are constant for a given four-bar mechanism. Coupler angle λ represents the varying orientation of coupler link with respect to fixed link at any given instant. Point P is the location of the coupler point in the global frame, which is also a variable. Coupler orientation ζ refers to the orientation of a moving frame attached to the coupler point, while δ is the constant angle at which moving frame is attached to coupler with respect to the coupler link line

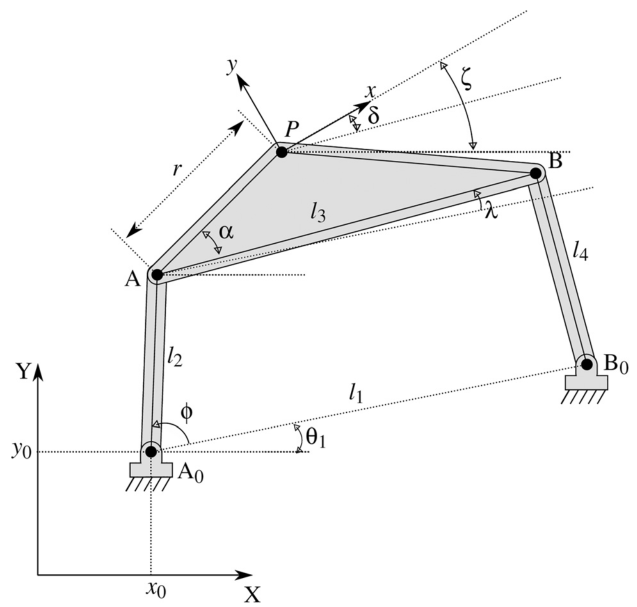

Fig. 2 Visualization of parameters describing a four-bar mechanism

AB. All of these design parameters are unknown before a mechanism has been synthesized. Our goal is to find an explicit closed-form relationship between coupler path and orientation which forms the heart of our mixed synthesis algorithm.

2.1 Coupler Angle.

The Fourier series representation of the coupler angle λ for a four bar mechanism is given as

$$e^{j\lambda} = \sum_{k=-\infty}^{\infty} C_k e^{jk\phi} = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega t} e^{jk\phi_0} \tag{1}$$

where C_k are the harmonic descriptors of coupler angle, ϕ the crank angle, ϕ_0 the initial crank angle, and ω is the constant angular speed of the input link.

2.2 Coupler Path.

The analytical equation which defines the path of coupler point P for a four bar mechanism is given by

$$P = A_0 + l_2 e^{j\theta_1} e^{j\phi} + r e^{j\alpha} e^{j\theta_1} e^{j\lambda} \tag{2}$$

where A_0 is the complex form of the position of input link fixed pivot, l_2 is the length of input link, θ_1 is the angle of fixed link, and r and α are the coupler parameters. Being a periodic function, it can also be represented as a Fourier series

$$P = \sum_{k=-\infty}^{\infty} P_k e^{jk\omega t} \tag{3}$$

Substituting Eq. (1) into Eq. (2) and then equating resulting (2) and (3), we get harmonic descriptors P_k for the path as follows:

$$P_0 = C_0 r e^{j(\alpha+\theta_1)} + (jy_0 + x_0); \quad k = 0 \tag{4}$$

$$P_1 = C_1 e^{j\phi_0} r e^{j(\alpha+\theta_1)} + l_2 e^{j\theta_1} e^{j\phi_0}; \quad k = 1 \tag{5}$$

$$P_k = C_k e^{jk\phi_0} r e^{j(\alpha+\theta_1)}; \quad k \neq 0, 1 \tag{6}$$

2.3 Coupler Orientation. The orientation (ζ) at the coupler point for a four-bar mechanism can be defined as

$$\zeta = \delta + \lambda + \theta_1 = \arg(e^{j(\delta+\lambda+\theta_1)}) \tag{7}$$

where δ is the fixed angle at which moving frame is attached to coupler with respect to $\theta_1 + \lambda$. As λ varies periodically while δ and θ_1 remain constant, the orientation can be decomposed harmonically as

$$e^{j(\delta+\lambda+\theta_1)} = \sum_{k=-\infty}^{\infty} C_k^* e^{jk\omega t} \tag{8}$$

where C_k^* are the harmonic descriptors for orientation and obtained as

$$C_k^* = C_k e^{j(\delta+\theta_1)} e^{jk\phi_0} \tag{9}$$

by substituting for $e^{j\lambda}$ from Eq. (1) in Eq. (8).

2.4 Path–Orientation Relation. With the above-mentioned relations, it is now possible to find explicit closed form relations between the Fourier descriptors of coupler path and coupler orientation data. Using Eqs. (4)–(6), and (9), relationship between the harmonic descriptors of path (P_k) and orientation (C_k^*) is found to be

$$C_0^* = (P_0 + z_2) z_1 \tag{10}$$

$$C_1^* = (P_1 + z_3) z_1 \tag{11}$$

$$C_k^* = P_k z_1 \tag{12}$$

where

$$z_1 = \frac{e^{j(\delta-\alpha)}}{r} \tag{13}$$

$$z_2 = -(x_0 + jy_0) \tag{14}$$

$$z_3 = -(l_2 e^{j\theta_1} e^{j\phi_0}) \tag{15}$$

Using the above-mentioned relationship, the orientation at coupler point can be defined exclusively using path harmonic descriptors as follows:

$$e^{j\zeta(t)} = z_1 \left(z_2 + z_3 e^{j\omega t} + \sum_{k=-\infty}^{\infty} P_k e^{jk\omega t} \right) \tag{16}$$

Subsequently, using Eq. (16) for n path points, the system of equation describing orientation at each path-point turns out to be

$$\begin{bmatrix} e^{j\zeta_1} \\ e^{j\zeta_2} \\ \vdots \\ e^{j\zeta_n} \end{bmatrix} = \begin{bmatrix} 1 & e^{j\omega t_1} & \sum_{k=p}^{-p} P_k e^{jk\omega t_1} \\ 1 & e^{j\omega t_2} & \sum_{k=p}^{-p} P_k e^{jk\omega t_2} \\ \vdots & \vdots & \vdots \\ 1 & e^{j\omega t_n} & \sum_{k=p}^{-p} P_k e^{jk\omega t_n} \end{bmatrix} \begin{bmatrix} z_1 z_2 \\ z_1 z_3 \\ z_1 \end{bmatrix} \tag{17}$$

Thus, the orientations at different points of a four-bar coupler path are dependent on path descriptors and three complex variables z_1, z_2, and z_3, which are termed as mixed synthesis parameters (MSP). The MSP are dependent on four-bar mechanism design parameters according to Eqs. (13)–(15). Equation (17) is the key to the mixed synthesis formulation. It will help us find orientation information for path points as discussed in Sec. 3.

3 Calculating Unknown Orientations

The aim of this section is to reformulate m-pose, n-path-point mixed synthesis problems into an $m+n$- pose motion synthesis problems. To enable that, generation of orientation data for n path points and converting them to n poses is required. Equation (17) will be used to accomplish this objective.

For a m-pose, n-path synthesis problem, a task path described by a trigonometric polynomial curve with an open interval is calculated and represented as

$$z(t) = \sum_{k=-p}^{p} T_k e^{ik\omega t} \qquad \forall \quad t \in [0, t_{\max}], t_{\max} < 1 \tag{18}$$

where $z(t) = x(t) + iy(t)$ denotes the point coordinates in complex form at time t, k are the frequency indices, T_k are the task curve Fourier descriptors, ω is the angular velocity of crank, and [0, $t_{\max}$] is the time interval over which the curve is defined. The T_k can be calculated by least square minimization of

$$\Delta = \sum_{i=1}^{n} \left\| z(t_i) - \sum_{k=-p}^{p} T_k e^{ik\omega t_i} \right\|^2 \tag{19}$$

where Δ is the fitting error measure and $z(t_i)$ are the complex-valued point data at time t_i. Analytically solving the minimization problem gives a linear system of equation as follows:

$$\mathbf{\Omega} \mathbb{X} = \mathbb{Y} \tag{20}$$

where

$$\mathbb{X} = [\ldots, \underset{m\rightarrow}{T_m}, \ldots]^{\mathrm{T}} \tag{21}$$

$$\mathbf{\Omega} = \left[\begin{array}{ccc} & \cdots & \\ \vdots & \sum_{i=0}^{n} e^{i(k-m)\theta_i} & \vdots \\ & \cdots & \end{array} \right] \downarrow m, \quad k\rightarrow \tag{22}$$

$$\mathbb{Y} = \Big[\ldots, \underset{m\rightarrow}{\sum_{i=0}^{n} z(t_i) e^{-im\theta_i}}, \ldots \Big]^{\mathrm{T}} \tag{23}$$

Here, k and m vary from $-p$ to p which denote the column and row index of an element in the matrix. Thus, $\mathbb{X}$ and $\mathbb{Y}$ are m-dimensional vectors while $\mathbf{\Omega}$ is a $m \times n$ dimensional matrix. Lower-upper decomposition can be used to solve the above-mentioned system. More details can be found in the work done by Wu et al. [14]. In Ref. [24], we have proposed a method to calculate optimal time parametrization for task curve for Fourier descriptor fitting of the path data. In our implementation, task curves are represented using up to eleven descriptors i.e., $p \in [-5]$. If $m+n < 11$, we use a lesser number of descriptors to generate a unique task curve.

The reasoning behind using a task curve with low higher order harmonic content is supported in literature [16,25], which says that the magnitude of high harmonics for coupler path of a four-bar mechanism has an insignificant impact. Thus, the fitted task path is a good prospective four-bar coupler curve and the task curve descriptors T_k can be equated to coupler path descriptors P_k.

The intention now is to find the MSP i.e., $\{z_1, z_2, z_3\}$ using available orientation data and subsequently generate unknown orientations. We define the system of equation given by Eq. (17) as fully constrained if all the MSP can be calculated exactly. For a fully constrained MSP computation problem, three poses are required to calculate the MSP directly from Eq. (17). Physically, this condition makes perfect sense as the user might know orientations at the initial position, final position and an additional intermediate location while a sequence of path points might be given in addition.

For under-constrained MSP computation problem, there are only one or two poses given. As a result, additional constraints are required to uniquely calculate the MSP. The MSP are dependent on four-bar mechanism parameters according to Eqs. (13)–(15). These equations can be used to generate additional constraints which are called mixed constraints (MIC). The three possible MIC are

(1) Specify coupler parameters i.e., $\{r, \alpha, \delta\} \rightarrow z_1$
(2) Specify actuating fixed pivot i.e., $\{x_0, y_0\} \rightarrow z_2$
(3) Specify scale of input link, orientation of fixed pivot line, and initial angle i.e., $\{l_2, \theta_1, \phi_0\} \rightarrow z_3$

Thus, if two poses are input by the user, one MIC is required to fully define the system of equations in Eq. (17). If only one pose is specified by user, two MIC are required to solve the problem. Two pose problems are fairly common when only the first and last orientations are important, such as in pick-and-place operations. The MIC also mirrors practical user-specified constraints, such as selection of the location of the fixed pivot where an actuator might be situated. In another case, there might be a restriction on coupler link dimensions. Thus, the MIC represents a set of practical design constraints.

It is important to note that a pure path synthesis problem cannot be restructured into a motion synthesis problem without fully defining all three MSP. However, constraining all MSP simultaneously makes the synthesis less useful as most of the mechanism parameters are then fixed.

For over-constrained MSP computation problems, the number of poses specified is more than three. In this case, a least square solution to Eq. (17) can be calculated using complex SVD. Real SVD solvers, which are more easily available, can also be used by reducing the complex system of equation in Eq. (17) into an equivalent real system of equation in accordance with Ref. [26]. The K_1 formulation presented in Ref. [26] has been used in our implementation. According to the formulation, a complex system of equation

$$(A + iB)(x + iy) = b + ic \tag{24}$$

can be written as a real system of equation

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix} \tag{25}$$

Finding least square solution to this equivalent real system of equations gives the solution to original complex problem and values of MSP can easily be calculated in over-constrained cases.

Once the values of MSP z_1, z_2, z_3 are calculated using m poses, orientations at n path points can be found out by simple matrix multiplication using the system of equation in Eq. (17). As a result, n path points and m poses are converted to $m+n$ poses. The motion synthesis algorithm can now be used to calculate dyads. A review of algebraic fitting-based motion synthesis algorithm is discussed in Sec. 4.

4 Motion Synthesis Algorithm

Now that the mixed synthesis problem has been reformulated as motion synthesis problem, solution mechanisms can be achieved by calculating the dyads. Algebraic fitting-based motion synthesis algorithm [7–10] has been used in our implementation. In this approach, a planar four-bar linkage is split open in two dyads and each dyad is computed independently thus reducing significant computational burden. Moreover, this approach enables us to carry out simultaneous type and dimensional synthesis of four-bar linkages, i.e., it takes into consideration the possibility of both revolute and prismatic joints. Another benefit of the approach is its fast and efficient computation.

First, using kinematic mapping [27], each of the user-defined pose $\{x, y, \zeta\}$ is mapped to quaternion space defined by a four-dimensional vector $\mathbf{Z} = \{Z_1, Z_2, Z_3, Z_4\}$ called planar quaternions [28]. This space is also termed as the Image Space of planar kinematics [27]. This mapping is defined by

$$Z_1 = \frac{1}{2}\left(x\cos\frac{\zeta}{2} + y\sin\frac{\zeta}{2}\right) \tag{26}$$

$$Z_2 = \frac{1}{2}\left(-x\sin\frac{\zeta}{2} + y\cos\frac{\zeta}{2}\right) \tag{27}$$

$$Z_3 = \sin\frac{\zeta}{2} \tag{28}$$

$$Z_4 = \cos\frac{\zeta}{2} \tag{29}$$

The geometric constraints of all types of dyads can be represented by a single algebraic equation as follows:

$$\begin{aligned} &q_1(Z_1^2 + Z_2^2) + q_2(Z_1Z_3 - Z_2Z_4) + q_3(Z_2Z_3 + Z_1Z_4) \\ &\quad + q_4(Z_1Z_3 + Z_2Z_4) + q_5(Z_2Z_3 - Z_1Z_4) + q_6Z_3Z_4 \\ &\quad + q_7(Z_3^2 - Z_4^2) + q_8(Z_3^2 + Z_4^2) = 0 \end{aligned} \tag{30}$$

where $q_i (i = 1, 2, \cdot, 8)$ are the homogeneous coefficients of the manifold surface represented by the above-mentioned equation. In Ref. [10], we call this as a generalized (G-) manifold, which is capable of representing all types of mechanical dyads. For every pose, one such linear equation with unknowns as q_i is obtained. Assembling all the G-manifold equations for all the poses results in the following over-constrained homogeneous linear system on equation:

$$\boldsymbol{A}\boldsymbol{q} = 0 \tag{31}$$

where

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & \cdots & \cdots & \cdots & A_{18} \\ A_{21} & A_{22} & A_{23} & A_{24} & \cdots & \cdots & \cdots & A_{28} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & A_{n3} & A_{n4} & \cdots & \cdots & \cdots & A_{n8} \end{bmatrix} \tag{32}$$

and

$$\boldsymbol{q} = [\, q_1 \quad q_2 \quad \cdots \quad q_8 \,]^{\mathrm{T}} \tag{33}$$

The elements of each row of the matrix $\boldsymbol{A}$ are given as

$$A_{i1} = Z_{i1}^2 + Z_{i2}^2 \tag{34}$$

$$A_{i2} = Z_{i1}Z_{i3} - Z_{i2}Z_{i4} \tag{35}$$

$$A_{i3} = Z_{i2}Z_{i3} + Z_{i1}Z_{i4} \tag{36}$$

$$A_{i4} = Z_{i1}Z_{i3} + Z_{i2}Z_{i4} \tag{37}$$

$$A_{i5} = Z_{i2}Z_{i3} - Z_{i1}Z_{i4} \tag{38}$$

$$A_{i6} = Z_{i3}Z_{i4} \tag{39}$$

$$A_{i7} = Z_{i3}^2 - Z_{i4}^2 \tag{40}$$

$$A_{i8} = Z_{i3}^2 + Z_{i4}^2 \tag{41}$$

where i is the pose index ranging from $i = (1, 2, \ldots, n)$. The least square solution to this homogeneous system of equation can be found out using the singular value decomposition of the coefficient matrix $\boldsymbol{A}$ [29]. The right singular vectors corresponding to the smallest singular values are candidate solutions for the minimization problem. The subspace spanned by the three smallest singular value right-vectors represents a family of possible dyad solutions. However, for these dyads to make physical sense, the following extra constraints are required to be satisfied:

$$\begin{aligned} q_1q_6 + q_2q_5 - q_3q_4 &= 0 \\ 2q_1q_7 - q_2q_4 - q_3q_5 &= 0 \end{aligned} \tag{42}$$

An analytical solution to the above reduces to a quartic equation, which can give zero, two, or four real dyad solutions. Complex solutions do not represent a physical dyad. Combining any of the two dyads results in a four-bar mechanism. For further details, see Ref. [10]. As a result, the path synthesis problem is solved and prospective solutions are generated. Using this motion synthesis algorithm also enables us to simultaneously carry out type and dimensional synthesis. However, that is not the focus of this work.

The above-mentioned methodology for motion computation works for five or more poses when the system of equation is fully constrained or over-constrained. To handle under-constrained cases, additional motion synthesis constraints (MOC) also called geometric constraints outlined in Ref. [8] are used. We note that the constraints being discussed here are the ones required to define the motion computation problem, which are different from the constraints discussed earlier in the context of MSP computation. We ensure that the context would make it clear which constraints are being discussed.

5 Unified Synthesis Algorithm

A unified synthesis algorithm to solve the Alt-Burmester problems has been summarized in Algorithm 1. It states that when the synthesis problem has zero poses, the Fourier descriptor-based path synthesis algorithm as described by Wu et al. [14] is used. For all the other cases, mixed synthesis approach using Eq. (17) can be used to solve for four-bar mechanisms. It must be noted that except for the case where there are no poses, the synthesis calculates both type and dimensions.

Algorithm 1. Algorithm for unified motion, path and mixed synthesis

Input: Path points and Poses
if $n(\text{Pose}) = 0$ **then**
  Calculate T_k using Eq. (19)
  Calculate the four-bar mechanism by solving a minimization problem in a four-dimensional subspace as described by Wu et al. [14]
else
  Calculate T_k using Eq. (19)
  Calculate MSP using Eq. (17)
  Calculate the singular vectors using Eq. (31)
  Calculate the dyads using Eq. (42)
end
Output: Synthesized mechanism

A key advantage of the methodology outlined is that it can handle motion, path, and mixed synthesis problems seamlessly. Various permutations of $(0, 1, \ldots, m)$ poses and $(0, 1, \ldots, n)$ path-point problems are presented in Table 1. The legends in the table are MOC = motion synthesis constraint [8], MIC = mixed synthesis constraint, FD = fully defined, and X = trivial or undefined. The "*" refers to conditions where a Fourier task curve with just four points needs to be fitted and would have unsymmetrical descriptors.

Table 1 Various possibilities for unified motion, path, and mixed synthesis problem

		Path points					
		0	1	2	3	4	n
Poses	0	X	X	X	X	FD*	FD
	1	X	X	X	2 MIC*	2 MIC	2 MIC
	2	X	X	1 MIC*	1 MIC	1 MIC	1 MIC
	3	2 MOC	1 MOC*	FD	FD	FD	FD
	4	1 MOC	FD	FD	FD	FD	FD
	5	FD	FD	FD	FD	FD	FD
	m	FD	FD	FD	FD	FD	FD

Motion synthesis constraints can be used to specify the position of fixed or moving pivots using line or point constraints or any other compatible geometric constraint; see Ref. [8] for details. Mixed synthesis constraints, described earlier, involve constraints on actuating pivot, coupler dimensions, and other mechanism parameters. Fully defined (FD) entails that no extra constraints are needed to exactly or least square solve the mixed synthesis Eq. (17). If either one of the MSP computation problem or motion synthesis problem is under-constrained, the mixed synthesis problem is defined to be under-constrained. These under-constrained cases in Table 1 require additional constraints to be solved. In the table, zero-pose (or, pure-path synthesis) problems are solved using Wu et al. [14]. In that case, when four or more path points are specified, the problem is fully defined and is non-trivial; however, for $n = 4$, we can only calculate unsymmetrical descriptors. When only one pose is given, then two MIC are required to calculate all the MSP; with two poses, one MIC is required; and for three poses, no additional MIC are needed. However, for three poses, at least two path points need to be specified to obtain five poses needed for the motion synthesis algorithm. Burmester [4] showed that one needs five poses to solve a motion generation problem uniquely. In Ref. [7], we have extended Burmester problem to show that one can specify not only five poses, but a combination of pose and other geometric constraints to have unique mechanism design solutions. Therefore, for one path point with three poses, we need one MOC and for zero path point, we need two MOC to get a total of five constraints. However, zero path-point problem reduces to a pure motion generation problem. We will illustrate some of these permutations and combinations in the examples next.

6 Examples

In this section, we present some examples to illustrate the effectiveness of the proposed algorithm. First example aims to validate the approach by extracting path points and poses from a known mechanism. Second example solves mixed synthesis problem with fully constrained MSP computation involving three poses and five path points. Third and fourth examples deal with under-constraint MSP computation and motion synthesis cases and require additional mixed- and motion-constraints, respectively. Demonstrating valid results from each of these cases proves the robustness of proposed algorithm. It also demonstrates the flexibility of the algorithm and its ability to incorporate various constraints. In Figs. 3–8, two curves representing coupler path for the two possible assembly modes have been displayed.

6.1 Example 1: Reverse Engineering a Mechanism.

To validate the proposed mixed synthesis algorithm, points and poses from a known planar four-bar mechanism are taken and then our algorithm is used to synthesize mechanisms. Ideally, we should get the exact same mechanism. However, a similar mechanism is also acceptable since approximations occur at various steps—from task curve generation to algebraic fitting of the pose data.

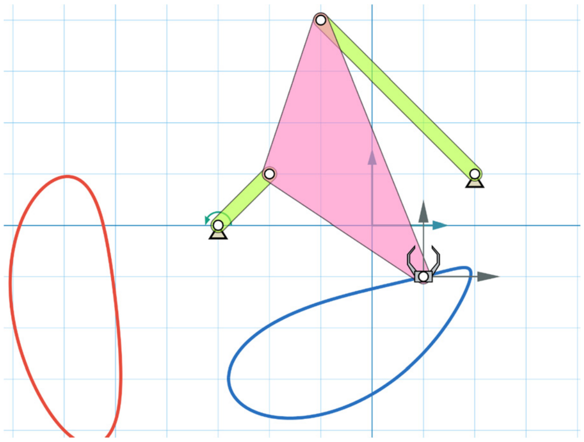

Fig. 3 Example 1: known target mechanism

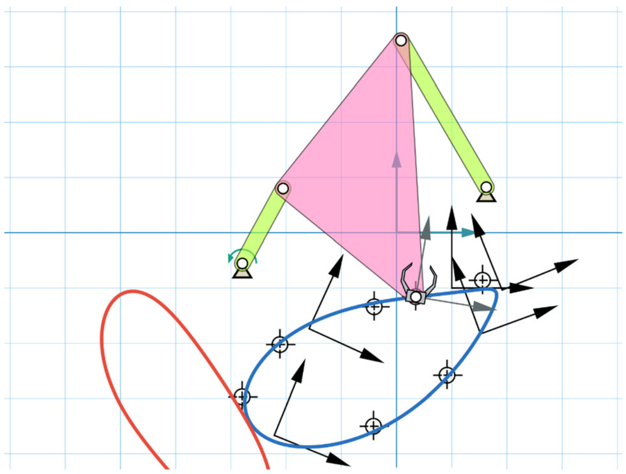

Fig. 4 Example 1: mechanism generated using mixed synthesis algorithm

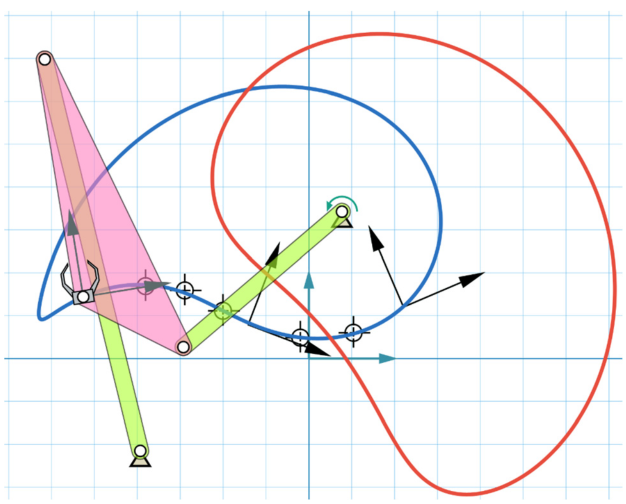

Fig. 5 Example 2: mixed synthesis with fully constrained MSP computation for three poses and five path points

A sample mechanism displayed in Fig. 3 is used to generate seven path points and five poses. The mechanism has been defined using the position of its fixed pivots, moving pivots and coupler coordinates as shown in Table 2.

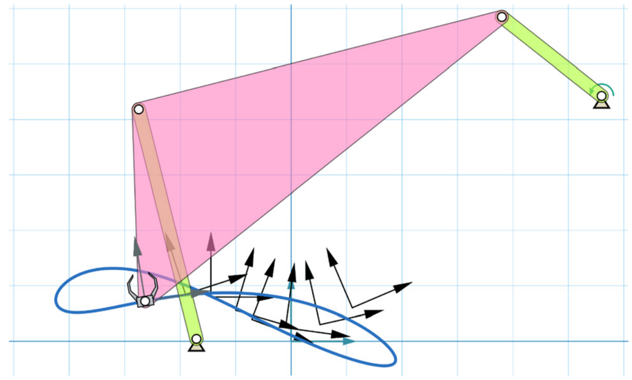

Fig. 6 Example 2: over-constrained motion synthesis for eight poses produces a poor solution

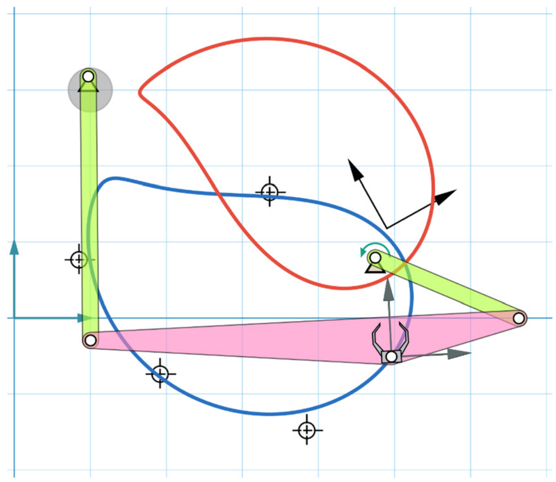

Fig. 7 Example 3: under-constrained mixed synthesis for two poses and four path points using additional mixed constraint

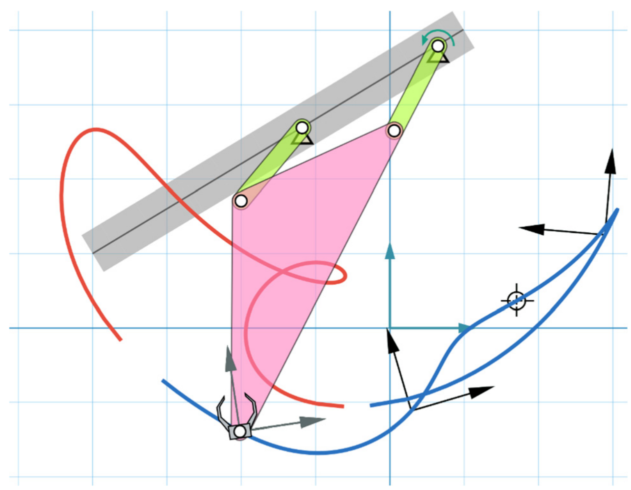

Fig. 8 Example 4: under-constrained mixed synthesis for three poses and one path points using additional motion constraint

The arbitrarily sampled poses and path points are listed in Table 3. This is a fully defined problem and all the MIC can be computed without requiring any additional information. These constraints are used as input to mixed synthesis algorithm. Four solution dyads are output as listed in Table 4. This also allows us to reverse-engineer a known mechanism since there are six planar

Table 2 Example 1: sample mechanism design parameters as shown in Fig. 3

Point	X	Y
Input link fixed pivot	– 3.0	0.0
Input link moving pivot	–2.0	1.0
Output link fixed pivot	2.0	1.0
Output link moving pivot	–1.0	4.0
Coupler point	1.0	–1.0

Table 3 Example 1: input data

No.	Type of data	x	y	ζ (rad)
1	Point	0.350	–1.160	—
2	Point	–0.410	–1.340	—
3	Pose	–1.585	–1.737	5.853
4	Point	–2.110	–2.030	—
5	Point	–2.800	–2.970	—
6	Pose	–2.216	–3.665	5.896
7	Point	–0.420	–3.500	—
8	Point	0.910	–2.580	—
9	Pose	1.520	–1.832	0.351
10	Pose	1.912	–1.036	0.385
11	Point	1.560	–0.860	—
12	Pose	1.000	–1.000	0.000

Table 4 Example 1: output dyad data

Dyad	Fixed pivot	Moving pivot	Coupler point
1	−2.793, –0.554	−2.056, 0.799	0.350, –1.160
2	−0.172, 6.978	1.340, 8.356	0.350, –1.160
3	−18.181, 11.280	−4.226, 6.257	0.350, –1.160
4	1.625, 0.824	0.081, 3.477	0.350, –1.160

four-bars that can satisfy the given constraints. Figure 4 shows a four-bar obtained by assembling dyads 1 and 4. It is observed that the mechanism generated is very similar to the original mechanism. This approximate result is due to the best-fitted low harmonic task curve following the original coupler curve closely but not exactly. The average path error measured by calculating the deviation of input points from final path using the Euclidean distance is 0.0694 units for the displayed configuration. The maximum angular deviation among all the given poses is for the pose 3 as 0.0736 rad.

In this example, the input data consisted of five poses and seven path points. Greater than three input poses over-constraints the MSP computation due to which the path–orientation relationship is satisfied using SVD. Thus, this example demonstrates mixed synthesis with over-constrained MSP computation.

6.2 Example 2: Mixed Synthesis With Fully Constrained Mixed Synthesis Parameters Computation. In this example, the input data consist of three poses and five path points which fully constrain the MSP computation problem. The data input to mixed synthesis algorithm is given in Table 5. The two dyads generated as output have been shown in Table 6. The final mechanism has been displayed in Fig. 5. It can be observed that a good match has been established with the constraints. Average path error is 0.0226 units while the maximum angular deviation for poses is 0.0106 rad for second pose. Note that in this case, the path–orientation relationship has an exact solution, i.e., MSP are uniquely determined using SVD.

One of the major advantages of mixed synthesis is the additional flexibility it imparts to users while specifying inputs and

Table 5 Example 2: input data

No.	Type of data	x	y	ζ (rad)
1	Pose	–5.263	1.441	0.161
2	Point	–3.810	1.690	—
3	Point	–2.890	1.590	—
4	Point	–2.010	1.120	—
5	Pose	–1.416	0.789	5.919
6	Point	–0.200	0.490	—
7	Point	1.040	0.600	—
8	Pose	2.206	1.203	0.405

Table 6 Example 2: output dyad data

Dyad	Fixed pivot	Moving pivot	Coupler point
1	0.771, 3.424	–2.927, 0.266	–5.263, 1.441
2	–3.931, –2.151	–6.160, 6.975	–5.263, 1.441

Table 7 Example 3: input data

No.	Type of data	x	y	ζ (rad)
1	Pose	4.962	–0.514	0.134
2	Point	3.850	–1.480	—
3	Point	1.920	–0.740	—
4	Point	0.850	0.760	—
5	Point	3.360	1.650	—
6	Pose	4.900	1.178	0.510

generating good solutions. Using a pure motion synthesis algorithm, the user would have to input all the data as poses even if the problem demanded otherwise. This would lead to an over-constrained motion problem, which when solved using existing kinematic mapping-based algebraic fitting approach [7–10] usually produces poor solutions. A comparable motion synthesis problem for the same path points, but with orientations also given is displayed in Fig. 6. It can be observed that the solution provides a poor fit to the given constraints. This happens because the orientation provided is not compatible with the motion of coupler of the planar-four linkages.

6.3 Example 3: Mixed Synthesis With Under-Constrained Mixed Synthesis Parameters Computation Using Mixed Constraints. This example shows mixed synthesis with under-constrained MSP computation problem where two poses and four path points are specified in the input. Lesser than three input poses makes MSP computation problem under-constrained. To solve for MSP, an additional mixed constraint is required which could specify any of z_1, z_2, z_3. The constraint data input to mixed synthesis algorithm is shown in Table 7. A MIC is used to specify z_2 by defining the preferred location of a fixed joint at point (1,3). The four dyads generated as output are shown in Table 8. One of the final mechanisms is displayed in Fig. 7 using dyads 3 and 4. It can be observed that the generated mechanism closely satisfies path and mixed constraints. The average path error is 0.0948 units while the maximum angular deviation for poses is 0.0389 rad for the second pose in the displayed mechanism. Note that in this case, the path–orientation relationship has an infinite solutions and the use of MIC restricts the solution space to a unique solution to the MSP.

6.4 Example 4: Mixed Synthesis With Under-Constrained Motion Synthesis Using Motion Constraints. This example shows mixed synthesis with under-constrained motion synthesis problem where three poses and one path point is specified in the

Table 8 Example 3: output dyad data

Dyad	Fixed pivot	Moving pivot	Coupler point
1	8.705, 10.738	8.705, 10.738	4.962, –0.514
2	–7.961, 8.082	–3.450, 6.381	4.962, –0.514
3	0.973, 3.179	1.004, –0.295	4.962, –0.514
4	4.748, 0.792	6.635, –0.004	4.962, –0.514

Table 9 Example 4: input data

No.	Type of data	x	y	ζ (rad)
1	Pose	–2.018	–1.391	0.146
2	Pose	0.288	–1.115	0.287
3	Point	1.700	0.360	—
4	Pose	2.895	1.253	1.487

Table 10 Example 4: output dyad data

Dyad	Fixed pivot	Moving pivot	Coupler point
1	0.647, 3.788	0.058, 2.651	–2.018, –1.391
2	–1.181, 2.692	–2.002, 1.708	–2.018, –1.391
3	–3.322, 1.407	–4.111, 3.061	–2.018, –1.391
4	–3.322, 1.407	–4.111, 3.061	–2.018, –1.391

input. Motion synthesis is under-constraint because the total pose and path constraints are just four. Even though three poses specified can be used to calculate the MSP, an additional motion constraint is required to solve the motion synthesis problem. The data input to mixed synthesis algorithm is given in Table 9. A line constraint is used as MOC in the example presented. The line segment is defined by its end points (–4,1) and (1,4). The four dyads generated as output are shown in Table 10. One of the final mechanisms is displayed in Fig. 8 using dyads 1 and 2. It can be observed that the generated mechanism closely satisfies path constraints. The average path error is 0.0026 units while the maximum angular deviation for poses is 0.0005 rad for second pose in the displayed mechanism. Also, both the fixed pivots fall on the line constraint specified. Thus, the synthesis problem is successfully solved. Note that in this case, it is not the path–orientation relationship that is under-defined but the algebraic fitting algorithm which requires at least five poses to be fully defined.

7 Conclusion

In this paper, we have presented a generalized m pose, n path-point mixed synthesis approach for four-bar mechanisms. Original contributions of this paper include the closed-form relationship between coupler orientation and coupler path and exploiting this relationship to present a novel framework for solving the mixed synthesis problem. Another novel feature is the use of task-driven motion synthesis algorithm within the framework to keep the computation cost at minimum and perform simultaneous type and dimensional synthesis. A few examples were presented to demonstrate the effectiveness of the approach.

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